(Based on Boyce-di Prima 10th ed.)

Canonical Forms and Solutions.

F.O.L. (general)	$P(t)y' + Q(t)y = G(t)$ * if it's in the above form, you have to rewrite it in the below form first. $y' + p(t)y = g(t)$	$y = \frac{1}{\boxed{\mu(t)}}\left[\int \boxed{\mu(s)} g(s)\, ds + c\right]$ (Integrating factor) $\boxed{\mu(t)} = e^{\int p}$ * note: the exponent is $\oplus$!! The expression $e^{-\int p}$ also comes up in the course, but that's for the Wronskian of a S.O.L. ODE.	(2.1)
F.O., n.n.L; seperable	$M(x)\,dx + N(y)\,dy = 0$ * it usually won't be in this form; you have to rearrange it into this format to determine if it's seperable	$\int M\,dx + \int N\,dy = c$	(2.2)
F.O., n.n.L; exact	$M(x,y)\,dx + N(x,y)\,dy = 0$ $\boxed{\text{and}}$ $M_y = N_x$	$(\psi :=) \underbrace{\int M\,dx}_{\text{"}H(x,y)\text{"}} + \int \underbrace{\left[N - \partial_y \int M\,dx\right]}_{\text{"}h'(y)\text{"}} dy = c$ * note: When doing $\int M\,dx$, $\boxed{\text{ignore}}$ the fn of y only that arises as the "constant" of integration. ((The second integral doesn't have this problem b/c it's a fn of y only))	(2.6)
F.O., n.n.L; n.n. exact	$M(x,y)\,dx + N(x,y)\,dy = 0$ (but $M_y \neq N_x$)	Solved w/ $\boxed{\text{integrating factors}}$. See $\boxed{\text{Addendum 1}}$ for how to $\boxed{\text{find}}$ an integrating factor. Once found, there's no general formula for the soln of the ODE in terms of μ; just multiply through by μ and then solve the resulting exact eqn. (as detailed on the prev. line)	(2.6)
F.O., n.n.L; autonomous	$y' = f(y)$	———————— (We only analyze these qualitatively / graphically)	(2.5)

S.O.L; C.C. + homog

$$ay'' + by' + cy = 0$$

Sol'n depends on roots of char. poly: (Ch.3)

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad (3.1)$$

$$y = c_1 e^{rt} + c_2 t e^{rt} \qquad (3.4)$$

$$y = e^{\lambda t}\left[c_1 \cos(\mu t) + c_2 \sin(\mu t)\right] \qquad (3.3)$$

S.O.L; N.C.C + homog.

$$y'' + p(t)\,y' + q(t)\,y = 0$$

$\boxed{\text{Reduction of Order}}$ (3.4)

$\boxed{\text{If}}$ some sol'n y_1 is known, then the other sol'n is

$$y_2 = y_1 \int \frac{W}{y_1^2}$$

W → (Wronskian) $:= y_1 y_2' + y_1' y_2 = ce^{-\int p}$ ↳ Abel's thm

(the first sol'n is often determined by guessing or some other insight)

S.O.L; inhomog. (coeff irrel.)

* literally any S.O.L ODE *

$\boxed{\text{Variation of Parameters}}$ (3.6)

$$y = y_c + \left[-y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W}\right]$$

y_c: sol'n to corresp. homog. eq'n.

* y_1 and y_2 are fund. sol'ns to the corresp. homog. eq'n.

$\boxed{\text{If}}$ (i) the ODE has const. coeff.
(ii) g has a special form, (sums/products of exp., poly's, and sin/cos.)

$\boxed{\text{Undetermined Coefficients}}$ (3.5)

Guess a particular sol'n (see $\boxed{\text{Addendum 2}}$ for the guesses) based on the inhomog term g. Determine coeffs and then add y_p to y_c, the sol'n to the corresp. homog. eq'n.

H.O.L; C.C. + homog.

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

Sol'n depends on roots of char. poly: (4.2)

- Each distinct (real) root contributes ce^{rt}
- Each complex (conj) pair contributes $e^{\lambda t}[c_1\cos(\mu t) + c_2\sin(\mu t)]$
- For roots w/ multiplicity, additional sol'ns are obtained by multiplying the "base" sol'n by increasingly higher powers of t (eg. te^{rt}, t^2e^{rt}, t^3e^{rt}, etc.)

H.O.L.; N.C.C + homog

$$p_n(t) y^{(n)} + \cdots + p_1(t) y' + p_0(t) y = 0$$

We don't do these b/c the best you can do is determine the last fund. sol'n if you already know the other (n-1). ((this is reduction of order)).

H.O.L; inhomog. (coeffs. irrel.)

↳ but in practice ours will always be C.C. since you still need to know a set of fund. sol'ns to the homog.

$$\left(a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(t)\right)$$

$\boxed{\text{Variation of Parameters}}$ (4.4)

$$y = y_c + \sum_{i=1}^{n} \int \frac{g W_i}{W}$$

y_c: sol'n to corresp. homog. eqn.

W_i: $\boxed{\text{signed}}$ minor of Wronskian matrix associated to the $W_{n,i}$ th entry (bottom row, ith column)

W: Wronskian

* for c.c. the wronskian will just be $e^{-(a_{n-1}/a_n)t}$ (by Abel's thm)

(unlike w/ S.O.L. eqns) this is a given for us when dealing w/ H.O.L.

$\boxed{\text{If}}$ (i) the ODE has const coeff

(ii) g has a special form

Same as for S.O.L.: sums/products of exp., poly's, and sin/cos.

$\boxed{\text{Undetermined Coefficients}}$ (4.3)

Exactly the same as the process for solving an (amenable) S.O.L.

F.O.L. system; C.C. + homog.

$$\underline{x}' = A\underline{x}$$

Sol'n depends on the spectrum of A (7.5, 7.6, 7.8)

- For 2x2 systems, see the $\boxed{\text{Phase Portrait summary.}}$
- For higher-dim'l systems, things become really complicated if there are eigenvalues w/ high algebraic multiplicity but low geometric multiplicity.

↳ Eigenvalues of alg. mult. 3 are treated in exercises 18 and 19 on p. 438

↳ We don't cover eigenvalues of higher multiplicity.

F.O.L. system; N.C.C. + homog.

$$\underline{x}' = P(t)\underline{x}$$

We don't cover this case.

fund. matrices are $\Psi = e^{\int P(t)}$

$\boxed{\text{special}}$ fund. matrix is $\Phi(t) = e^{\int_{t_0}^{t} P(t)}$

↳ I.C. $(t_0, \underline{e}_i)$ for $\underline{x}^{(i)}$

F.O.L. system; inhomog. (coeffs. irrel.)

↳ but in practice ours will always be C.C. since you still need to know the soln to the homog.

$$\underline{x}' = P(t)\underline{x} + \underline{g}(t)$$

$$\left(\underline{x}' = A\underline{x} + \underline{g}(t)\right)$$

$\boxed{\text{Variation of Parameters}}$ (7.9)

$$\underline{x} = \Psi(t)\cdot\underline{c} + \Psi(t)\int \Psi^{-1}(s)\cdot\underline{g}(s)\,ds$$

soln to corresp. homog. eqn ($\underline{x}^c$)

* $\Psi(t)$ is $\boxed{\text{any}}$ fund. matrix for the corresponding homog. eqn *

The soln to the $\boxed{\text{IVP}}$ through $(t_0, \underline{x}^0)$ can be obtained by integrating $\boxed{\text{from } t_0}$ to t and taking $\underline{c}^0 = \Psi(t_0)^{-1}\cdot\underline{x}^0$

(F.O.L. system; inhomog $\boxed{\text{continued}}$)

$$\left(\underline{x}' = P(t)\underline{x} + \underline{g}(t)\right)$$

$\boxed{\text{If}}$ the system has const. coeff.

P(t) is a const. matrix A

(like w/ H.O.L. single equations), this is a given for us when dealing w/ systems.

$\boxed{\text{If}}$ (i) the system has const. coeff.

(ii) g has a special form

$\boxed{\text{Diagonalization}}$ (7.9)

Changing coords to $\underline{y} = P\underline{x}$, where $P^{-1}AP$ diagonalizes A, the system becomes:

$$\underline{y}' = D_A \underline{y} + P^{-1}\underline{g}(t).$$

This is just a set of $\boxed{\text{uncoupled}}$ F.O.L. ODE's:

$$\left\{ y_i' = \lambda_i y_i + (i^{th} \text{ comp. of } P^{-1}g) \right.$$

Solving as normal, we then have $\underline{x} = P^{-1}\underline{y}$

$\boxed{\text{Undetermined Coefficients}}$ (7.9)

(Based on Tennenbaum + Pollard)

Sept. 23rd, 2019.

Addendum 1
Integrating Factor Summary.

1. Integrating factors are applicable to eqn's of the form:

$$P(x,y)\,dx + Q(x,y)\,dy = 0.$$

2. There are five types to search for first:

Function of...	Auxilary fn F	Integrating factor h
x only	$\dfrac{P_y - Q_x}{Q}$	$e^{\int F(x)\,dx}$
y only	$\dfrac{Q_x - P_y}{P}$	$e^{\int F(y)\,dy}$
xy	$\dfrac{P_y - Q_x}{yQ - xP}$	$e^{\int F(xy)\,d(xy)}$
x/y	$\dfrac{y^2(P_y - Q_x)}{xP - yQ}$	$e^{\int F(x/y)\,d(x/y)}$
y/x	$\dfrac{x^2(Q_x - P_y)}{xP - yQ}$	$e^{\int F(y/x)\,d(y/x)}$

3. The solution to the original eqn. is the same as the solution to the "augmented" eqn. (i.e. the equation that you get after multiplying through by the integrating factor).

4. The family of solutions is $f(x,y) = c$, where f is the primitive of the form $P\,dx + Q\,dy$.

(based on Table 3.5.1 on p. 182 of Boyce–di Prima)

Addendum 2.

Ansatz for Undetermined Coeffs.

1. Check that $g(t)$ is a sum of terms of the form

$$e^{\alpha t} P_n(t) \cos(\beta t) / \sin(\beta t)$$

* note: each term of $g(t)$ may contain only $\boxed{\text{one}}$ of sin and cos (but different terms may contain the other)

2. The component of y_p corresponding to each term in $g(t)$ is determined separately.

The ansatz are:

$e^{\alpha t}$	$c t^s e^{\alpha t}$
$P_n(t)$	$t^s Q_n(t)$
$\cos(\beta t) // \sin(\beta t)$	$t^s [c_1 \cos(\beta t) + c_2 \sin(\beta t)]$
$e^{\alpha t} P_n(t)$	$t^s Q_n(t) e^{\alpha t}$
$e^{\alpha t} \cos(\beta t) // \sin(\beta t)$	$t^s e^{\alpha t} [c_1 \cos(\beta t) + c_2 \sin(\beta t)]$
$P_n(t) \cos(\beta t) // \sin(\beta t)$	$t^s [Q_n(t) \cos(\beta t) + \tilde{Q}_n(t) \sin(\beta t)]$
$e^{\alpha t} P_n(t) \cos(\beta t) / \sin(\beta t)$	$t^s e^{\alpha t} [Q_n(t) \cos(\beta t) + \tilde{Q}_n(t) \sin(\beta t)]$

↳ this one basically subsumes all the others (just omit appropriate factors)

<u>Note 1:</u> although each term of $g(t)$ may only ever contain cos $\boxed{\text{or}}$ sin, the corresponding ansatz always contains $\boxed{\text{both}}$

<u>Note 2:</u> The factor "t^s" is present to ensure that the ansatz is linearly independent from the fund. sol'ns to the homog. eq'n. Choose s as small as possible ($s=0$ is ok) to make that the case.

Phase Portraits – Homog. 2x2 F.O.L. systems w/ const. coeffs. ((real))

① EVs real and distinct, opposite signs. → $\boxed{\text{saddle pt. @ } \underline{0}}$ — always unstable.

$$\boxed{\underline{X} = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2}$$

Bifurcation ② One EV = 0, the other nonzero ((real)) → $\boxed{\text{crit. line}}$ (through $\underline{0}$ and ξ_λ)

$$\boxed{\underline{X} = c_0\xi_0 + c_\lambda e^{\lambda t}\xi_\lambda}$$

③ EVs real and distinct, same sign → $\boxed{\text{proper node}}$ @ $\underline{0}$

$$\boxed{\underline{X} = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2}$$

→ same expression as case ①

Bifurcation ④ One real EV w/ mult. 2

[4a] ↳ matrix is not defective (i.e. two L.I. EVs) → $\boxed{\text{Star-node}}$ @ $\underline{0}$ ("semiproper"?)

$$\boxed{\underline{X} = e^{\lambda t}\underline{v}}$$

→ $\underline{v}$ is literally any vector in $\mathbb{R}^2$; this expression is a degenerate form of that in ①/③ when $\lambda_1 = \lambda_2$

entails one degenerate case: $\lambda = 0$ (x2)

[4b] ↳ matrix is defective (only one eigenvector) → $\boxed{\text{improper node}}$ @ $\underline{0}$

$$\boxed{\underline{X} = (c_1 + c_2 t)e^{\lambda t}\xi + c_2 e^{\lambda t}\eta}$$

→ ξ is the eigenvector, and η is the generalized eigenvector (which is guaranteed to exist)

⑤ EVs complex ((conj)), Re ≠ 0 → $\boxed{\text{spiral point}}$ @ $\underline{0}$

$$\underline{X} = e^{\lambda t}\left[c_1\cos(\mu t) + c_2\sin(\mu t)\right]\underline{a} + e^{\lambda t}\left[c_2\cos(\mu t) - c_1\sin(\mu t)\right]\underline{b}$$

$$= e^{\lambda t}\left[\underline{a} \mid \underline{b}\right]\begin{bmatrix}\cos & \sin \\ -\sin & \cos\end{bmatrix}\begin{bmatrix}c_1 \\ c_2\end{bmatrix}$$

→ eigenvals are $\lambda \pm \mu i$; eigenvectors are $\underline{a} \pm \underline{b}i$, where $\underline{a}, \underline{b} \in \mathbb{R}^2$.

Fund. sol'ns are:
$\underline{u} = e^{\lambda t}\left[\cos(\mu t)\underline{a} - \sin(\mu t)\underline{b}\right]$
$\underline{v} = e^{\lambda t}\left[\cos(\mu t)\underline{b} + \sin(\mu t)\underline{a}\right]$

Cases ②–⑤: can be either unstable or asymptotically stable. character depends on the sign of: ((⊕ ⇒ unstable, ⊖ ⇒ A.S.))

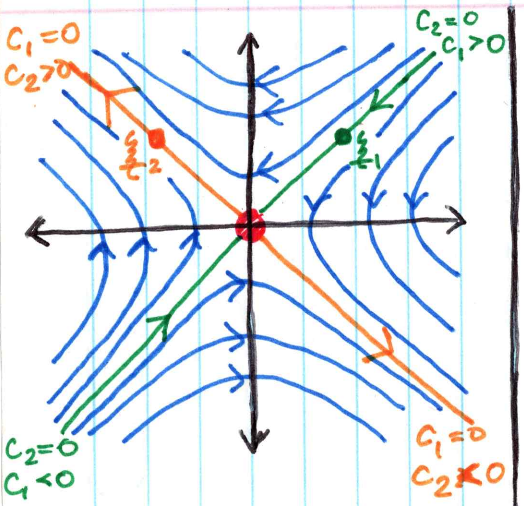
②	λ
③	λ_1 // λ_2
[4a]	λ
[4b]	λ
⑤	Real part

Bifurcation ⑥ EVs complex ((conj)), Re = 0 (i.e. purely imag.) → $\boxed{\text{centre}}$ @ $\underline{0}$. — always stable.

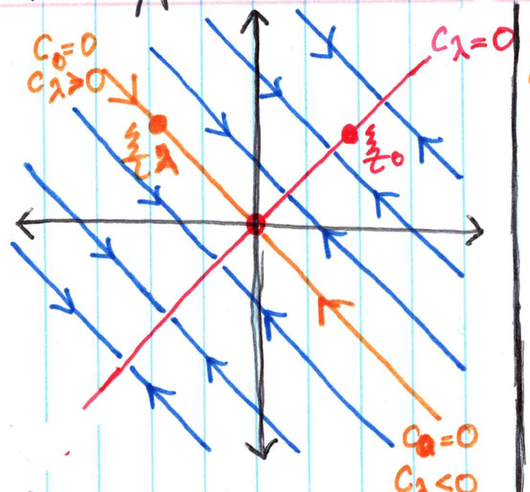
↳ between stable spiral and unstable spiral.

$$\boxed{\underline{X} = (\text{same as ⑤ but w/o } e^{\lambda t} \text{ in front})}$$

Prototypical Phase portraits.

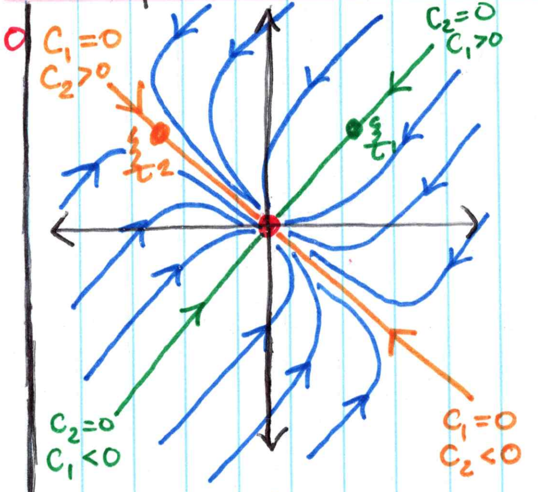


① Saddle point



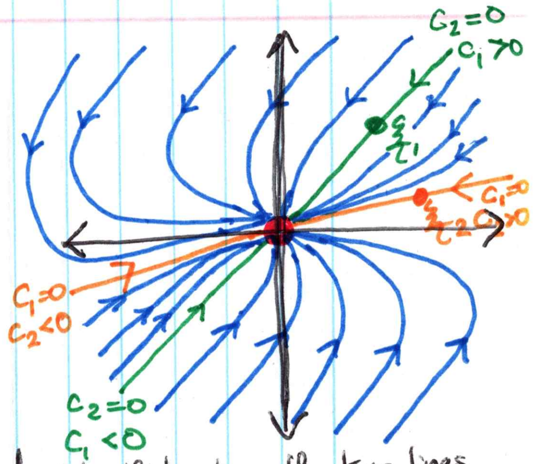
② ((no formal name))

↳ this is a degenerate case

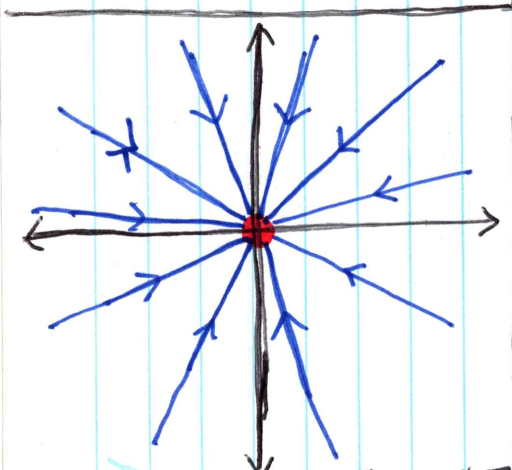


③ Proper node.

"Tails" parallel to one of the distinguished lines, and ~~[illegible]~~ near zero they curve to be parallel to the other. The tails are parallel to the line corresp. to the ~~bigger~~ EV w/ larger abs. value. Above would have $|\lambda_1|>|\lambda_2|$



Note that when the two lines form a small angle, the non-distinguished solns start to curve in a way that resembles the behaviour around an improper node (④)



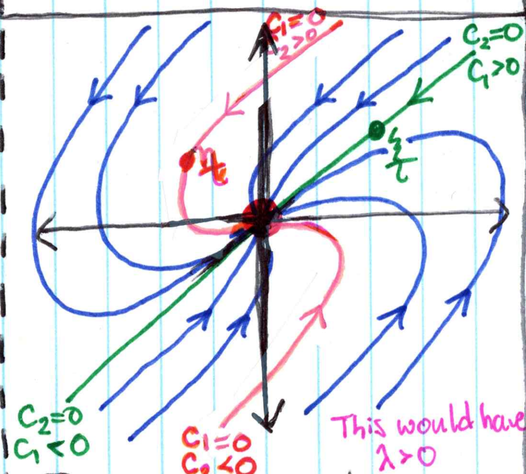
④a Star node [technically a sub-type of proper node]

No distinguished soln's

To draw the special soln through $\vec{\eta}$ (orange), just look @ the sign of λ:

⊕ ⇒ tail goes along $+\vec{\xi}$-ray

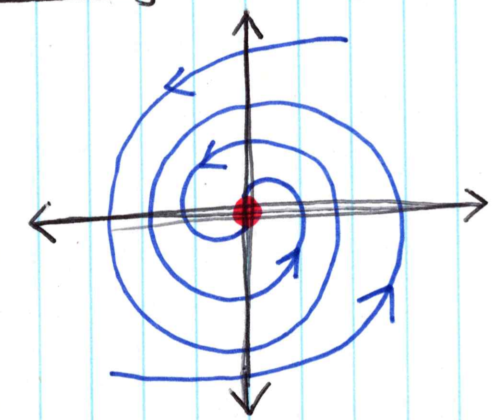
⊖ ⇒ tail goes along $-\vec{\xi}$-ray.



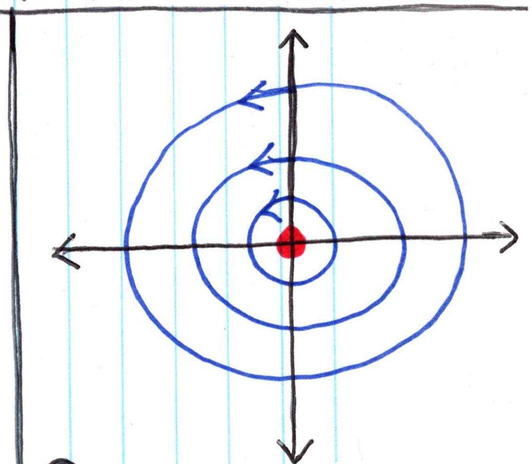
④b Improper node

Only one distinguished soln is a line; the other (orange) is curved.

Tails are parallel to the straight line, and then they curve back to arrive @ $\vec{0}$ parallel to it once again. see Addendum 1 for degenerate case.



⑤ Spiral point



⑥ Centre.

For both of these cases, there are no distinguished soln's. To determine the direction of rotation, look @ the imaginary part:

⊕ ⇒ CW

⊖ ⇒ CCW

[See BDP 10 pp. 501-502]

Note: the trajectories in cases 5 and 6 can be skewed towards ellipses, as opposed to just perfectly circular as shown above.

Addendum 1: Improper Node (4b)

Degenerate case

Consider the case where $\lambda = 0$ (x2). As long as there's only one eigenvector, this case isn't $\boxed{\text{totally}}$ degenerate.
((if it has two L.I. eigenvectors, then the matrix is the zero matrix)).

For example, take $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has $\underset{\sim}{\xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underset{\sim}{\eta} = \begin{bmatrix} X \\ 1 \end{bmatrix}$.

Notice that every matrix falling into this category ($\lambda = 0$ (x2), deficient) has A_0 as its Jordan Canonical Form.

Hence the phase portrait of every such matrix is just the one of A_0 under a change-of-coord.

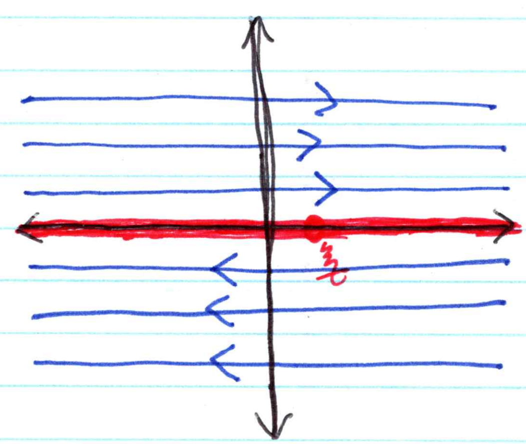
A_0 can be understood w/o writing down the general sol'n explicitly.

Note that $\underset{\sim}{\dot{x}} = A_0 \underset{\sim}{x}$ is just $\begin{cases} x' = y \\ y' = 0. \end{cases}$

- $y' = 0$ tells us that sol'n curves are purely horizontal
- $x' = y$ tells us that the direction (and speed) that the trajectories are traversed depends on their height.
- equilibrium points are the entire x-axis (($y = 0$)).

To find out exactly how another given system of this type looks, you do have to write down the general sol'n:

$$(c_1 + c_2 t)\underbrace{\underset{\sim}{\xi}}_{\text{determines red line}} + c_2 \underset{\sim}{\eta}$$



Adendum 2: Direction & Rotation (for 5 + 6)

Any 2x2 real matrix $\bar{w}$ complex (conj) EV's $\lambda \pm \mu i$ is similar to the canonical form $\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$.

In polar coords, the system $\dot{\underline{x}} = A\underline{x}$ has solns $\begin{cases} r = r_0 e^{\lambda t} \\ \theta = \theta_0 - \mu t \end{cases}$

Hence $\mu = (+) \Rightarrow \theta$ decreases $\Rightarrow$ CW rotation

$\mu = (-) \Rightarrow \theta$ increases $\Rightarrow$ CCW rotation.

Examples of matrices for each case.

1 $\begin{bmatrix} & \\ & \end{bmatrix}$

2 $\begin{bmatrix} & \\ & \end{bmatrix}$

3 $\begin{bmatrix} & \\ & \end{bmatrix}$

4a $\begin{bmatrix} & \\ & \end{bmatrix}$

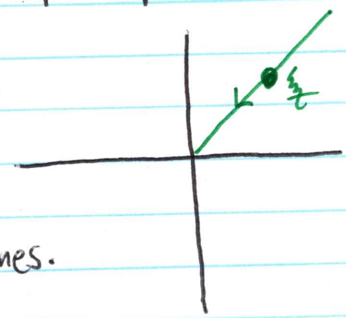
4b $\begin{bmatrix} & \\ & \end{bmatrix}$

4b- degenerate $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda = 0$ (x2) $\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\eta = \begin{bmatrix} x \\ 1 \end{bmatrix}$

5 $\begin{bmatrix} & \\ & \end{bmatrix}$

6 $\begin{bmatrix} & \\ & \end{bmatrix}$

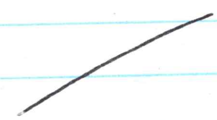
Improper node. (how formula leads to phase portrait)



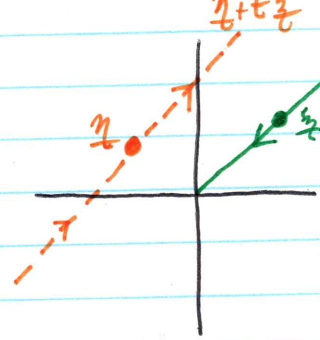
1. Consider $c_2 = 0$, $c_1 = 1$. ~~The special soln is~~
The corresp. soln is $e^{\lambda t}\xi$.

This manifests like all other cases w/ distinguished lines.

2. consider $c_1 = 0$, $c_2 = 1$.
The corresp. soln is $e^{\lambda t}(\eta + t\xi)$.
Think of this as a line w/ variable PW scaling.



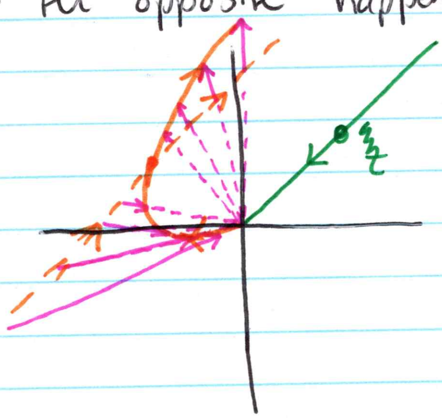
3. Draw the line $\eta + t\xi$ (dotted, since it's not an actual soln).
Notice that ~~the direction~~ it's parallel to the first special soln, but the <u>direction may be different</u>

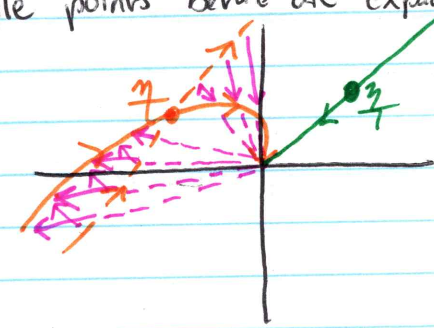


4. Apply the PW scalings $e^{\lambda t}$.
   - Since η is the basepoint of the line, $e^{\lambda t}$ reduces to $e^{\lambda \cdot 0} = 1$, so it doesn't move.

If $\lambda < 0$. • points after η are scaled in, while points before are expanded out.

If $\lambda > 0$ • the opposite happens.





$\lambda < 0 \Rightarrow$ come in parallel to the $-\xi$ ray

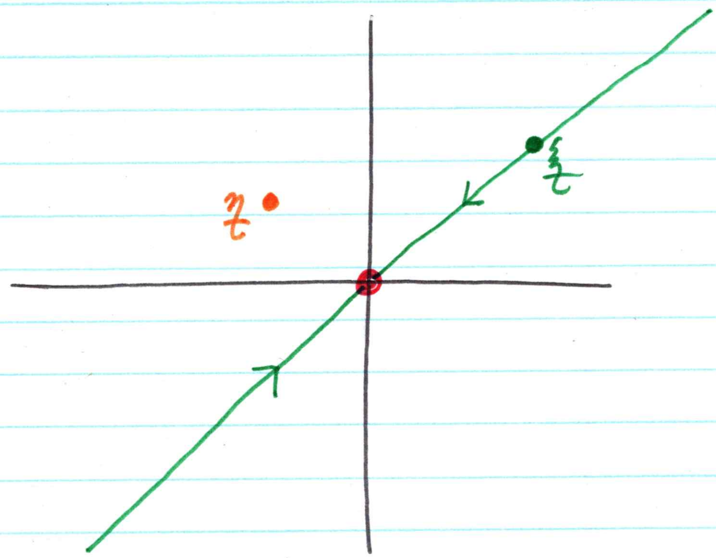
$\lambda > 0 \Rightarrow$ go out parallel to the $+\xi$ ray.

$\boxed{\text{Basically}}$ look @ sign of λ. ~~If~~ ⊕ the orange special soln passes through η going <u>out</u>. ⊖ $\Rightarrow$ passes through η going in.

Then just fill in all others.

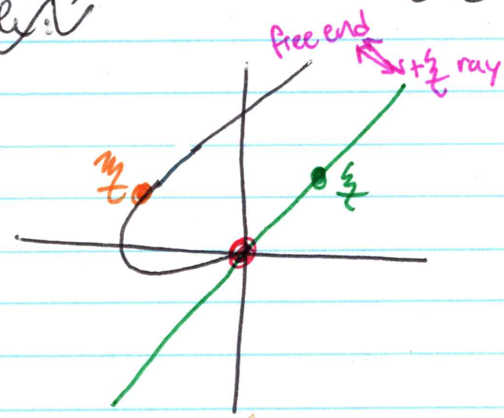
How to draw the Phase Portrait for an improper node.

1. Put. eq. pt, special ~~solns~~ line corresponding to ξ $((c_2=0))$ and η.

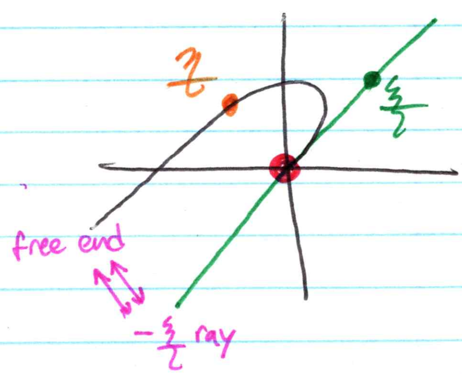


The tail of the special soln through η comes in with: ~~on the η side of the green line, the "free" ends of the soln curves [illegible] the~~ come in with.

$\lambda = \oplus \Rightarrow$ $+\xi$ ray:



$\lambda = \ominus \rightarrow$ $-\xi$ ray:



Fill in the rest and add arrows according to $\text{sgn}(\lambda)$.

Nonlinear F.O. systems (9.2)

qualitatively w/ drawing graphs

- Just like we were able to treat certain nonlinear first-order single equations,

those that are autonomous

again, $\boxed{\text{autonomous}}$

we can also treat certain nonliner F.O. systems, by drawing phase portraits.

- Notice that the classification axes "const/nonconst. coeff" and "homog/inhomog." are absent from the above discussion.
  - In a nonlinear context, the concept of a coefficient doesn't even really exist
  - In a nonlinear context, homogeneity is pretty much irrelevant. In the linear context, the point of the distinction is that the homog. is easy to solve, and then we can pull the soln of the inhomog. out of the soln to the homog. In the nonlinear case, there's often no such piggy-backing, so the distinction is irrelevant.

    ↳ Consider $(x+1)(x-2)=6$ vs. $(x+1)(x-2)=0$; knowing the soln to the latter ((homog.)) does nothing toward solving the former ((inhomog.)); we just have to re-solve it in full.

- Also, just like w/ linear systems, we only do the 2x2 case since that's all that we can draw.

9.3 - Local linearization of certain FOA 2x2 systems.

- Stability of equation $\underline{x}' = A\underline{x}$ under perturbation of coeffs

All characters of systems are stable under small perturbations of A $\boxed{\text{except}}$ spiral points and nodes.
The two exceptions are unstable b/c they correspond to bifurcations already (recall what ~~the~~ eigenvalues cause them), so a perturbation of A will tip the scale one way or the other.

Linearization: @ $\underline{x}^\circ$

1. change to $\underline{u} := \underline{x} - \underline{x}^\circ$ so that crit pt. is @ origin.

2. Search for $A, \underline{g}$ s.t. $\underline{f}(\underline{u}) = \underbrace{A\underline{u}}_{\text{linear approx.}} + \underbrace{\underline{g}(u)}_{\text{error.}}$
   a) Try the suff. cond. below.
   b) find A by taking linear terms of ~~the~~ $\underline{f}$.
   c) Might have to finesse a bit to make A invertible (add some linear terms, and then subtract them again; absorb error into $\underline{g}$). see Ex. 2 on p. 521 & BDP.

(want $\|g\|/\|u\| \to 0$ as $\|u\| \to 0$.)

$\Longleftrightarrow g_1/r,\ g_2/r \to 0.$

To check this, it's helpful to write g_1, g_2 in polar form.

~~Necessary~~ sufficient condition: F, G are C^2 near $\underline{x}^\circ$.

Then: $\dot{\underline{u}} = D\underline{f}_{\underline{x}^\circ}(\underline{u}) + \underline{\eta}(\underline{u})$

HOT from Taylor series for F, G.

$((\eta_i / r_{(u)} \to 0.))$