(s-)Injectivity of the attenuated geodesic x-ray transform of symmetric covariant tensor fields on negatively curved manifolds

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Abstract

The purpose of this report is to expound the proof of the landmark result presented in *Carleman Estimates for Geodesic X-ray transforms* by Gabriel Paternain and Mikko Salo from 2018. The abstract of that paper is quoted below for convenience. Their result follows 40 years of work concerning inverse problems in tensor tomography. This report is targeted both at students, in that it attempts to work out all details and aims to provide as much insight as possible, and at specialists, being written from a top-down POV and providing many reference materials (in appendices).

> In this article we introduce an approach for studying the geodesic X-ray transform and related geometric inverse problems by using Carleman estimates. The main result states that on compact negatively curved manifolds (resp. nonpositively curved simple or Anosov manifolds), the geodesic vector field satisfies a Carleman estimate with logarithmic weights (resp. linear weights) on the frequency side. As a particular consequence, on negatively curved simple manifolds the geodesic Xray transform with attenuation given by a general connection and Higgs field is invertible modulo natural obstructions. The proof is based on showing that the Pestov energy identity for the geodesic vector field completely localizes in frequency. Our approach works in all dimensions ≥ 2, on negatively curved manifolds with or without boundary, and for tensor fields of any order.

Preface

This report serves three functions, which also illuminate its intended audience

1. To record what I learned while working on this project and to demonstrate my understanding
2. To help my supervisor understand the paper. He and other specialist/expert readers will be most interested in my commentary on the high-level structure of the paper and its proofs, since I don't think any of the math is hard to follow if you're familiar with the territory. Appendix A contains all this kind of material. The other aspect that I think such readers will appreciate are all the nuances/distinctions that I've pointed out in places where things are not clear in the paper.
3. To help future students have an easier go at learning the paper. They will appreciate appendix C, since I've attempted to collect all the prerequisite knowledge into one place. Although I haven't expounded all of it, I (hope that I) at least recorded what one should learn and where to find it.

This report is not entirely a stand-alone item; it's more of a companion to the paper itself. I've retained the numbering of results, and this report makes many references back to the original text. This report follows one thread within the larger paper: specifically, how the Carleman estimate is used to give a partial solution to the injectivity problem for ray transform. Although the estimate or other results along the way may be of independent interest, I've generally ignored that in order to make this report more focused. As well, this report only follows the development of one of the two Carleman estimates given in the paper – this report doesn't address Theorem 1.2 and Chapter 7.

Literature review

There are essentially four papers that lead up to [PS18]: [PSU12, PSU13, PSU15, GPSU16]. [GPSU16] helps quite a bit with understanding [PS18]; I would recommend that students read it first because there are certain things that are completely omitted from PS18 (e.g. the discussion around s-injectivity). The other one that I would recommend reading is [PSU12], because it lays down the preliminaries for connections and Higgs fields; the rest of the papers assume this stuff tacitly. It should be noted, though, that as with all the other work from the early part of the decade, the paper only treats the 2D case, so some modifications are necessary.

How to read this report

- This report uses mostly the same notation as in the paper, but I've made a few changes to improve clarity. Where changes were made, I pointed them out
- The knowledge that I'm assuming of anyone who wants to use this report to understand PS18 is just a good grasp of smooth manifolds and Riemannian Geometry. All the other little bits (and there are a lot) are included in appendix C

Some peculiarities of my writing style

- I use parentheses, brackets, and multiplication dots to break up formulae to improve readability. If you find yourself wondering "why is there a dot here and not there; do they mean different things, etc.", the answer is almost certainly no - it's just for aesthetic reasons. If you really want to be sure, ask yourself how much harder it would be to read/parse if it were written in a more consistent manner/how you think it should be written
- I put most hypotheses and notation as a bulleted list at the end of a theorem statement, so that the text itself highlights the main point as clearly as possible
- I have the somewhat non-standard practice of including page numbers with all references. Mostly it's for my own benefit; I double/triple/quadruple check myself and I'm constantly going back to get additional details, etc, and it's much more efficient than having to hunt for the item each time. I intended to include the exact edition of each item used in the bibliography, but I wasn't able to figure out how to get Mendeley and bibTex to play nice, so that'll have to wait until the next version
- I start each new section on its own page because I mark up my texts so heavily that otherwise the section headings basically disappear into the background and I can't find my place

Contents

Chapter 1

Introduction

1.1 The Geodesic X-Ray Transform

The simplest version of the ray transform is the Radon transform, which takes a function on $\mathbb{R}^2$ and returns the family of values obtained by integrating it over all lines L:

$$R[f](L) \coloneqq \int_L f$$

The attenuated Radon transform includes a cumulative factor - known as the attenuation) that scales f before integrating:

$$R[f](L) \coloneqq \int_L \exp\left(\int_{L(-\infty,t]} a\right) \cdot f$$

There are many directions in which to generalize:

- of course, we can move to higher dimensions
- we can consider the geometric domain to be a manifold rather than $\mathbb{R}^n$ (in which case we integrate over geodesics)
- we can consider other objects that can be integrated, such as tensor fields

The case that we're interested in is symmetric covariant tensor fields on a certain class of manifolds:

Definition. A **CDRM** (compact dissipative Riemannian manifold) is a manifold with the following properties:

(i) with boundary
(ii) compact
(iii) dissipative/non-trapping (no infinite geodesics)
(iv) boundary is strictly convex (geodesics between two boundary points are entirely contained in M°; more on this in C.2.1)

Let's now discuss why this class of manifolds is the appropriate setting. The first thing to note is that the problem arises as the linearization (in g) of the problem of determining a metric on a manifold with boundary from its hodograph. So that justifies the restriction to manifolds with boundary.

To define the ray transform of a tensor, we need to define what we mean by integrating a tensor along a curve:

Definition. Given a tensor field T on M, we "lift" it to a function on SM by evaluating it in a specific way: for $\theta = (x, \underline{v}) \in SM$ we define $\widetilde{T}(\theta) \coloneqq T_x(\underline{v}, \cdots, \underline{v})$ (i.e. plugging $\underline{v}$ into all the slots).

Here's what the attenuated ray transform looks like on a manifold:

$$I^{\mathcal{A}}[T](\theta) = \int_0^{\tau(\theta)} \exp\left(\int_0^t \mathcal{A}\left(\phi_s^X(\theta)\right)\, ds\right) \cdot \widetilde{T}\left(\phi_t^X(\theta)\right)\, dt$$

where:

- T is a covariant tensor field
- $\widetilde{(\cdot)}$ is the evaluation/lifting operation
- θ is the general point of $\partial_+ SM$ (the inward-pointing boundary vectors)
- ϕ_t^X is the geodesic flow: $\phi_t^X(\theta) = (\gamma_\theta(t), \dot{\gamma_\theta}(t))$
  - X is the geodesic vector field
  - γ_θ is the geodesic determined by the initial condition $\theta = (x, \underline{v})$
- $\mathcal{A}$ is the attenuation
- $\tau(\theta)$ is the time at which the geodesic γ_θ terminates (by arriving at ∂M)

Note: we write the attenuation factor first so that the expression still makes sense in the case that T is vector-valued; in that case, $\mathcal{A}$ is a matrix, so we interpret exp as the matrix exponential, and the dot as matrix multiplication.

When we study the ray transform, we find it helpful to work with a version that's defined on all of SM rather than just $\partial_+ SM$. It's denoted u_T, and it's defined by the same formula by just letting θ vary over all of SM.

It should be noted that u_T is in general not smooth on $\partial_0 SM$; it inherits its non-smoothness from $\tau(\theta)$ [Sha94] (p.128). However, Prop 5.2 of [PSU12] (bottom of p.16) tells us that it's smooth for $T \in \ker I$. Note: the proof piggybacks off Lemma 1.1 of [PU05] (p.7; proof on p.14), which refers to a result in a PDE text by Hormander.

A fundamental fact in the study of the ray transform is that $(X + \mathcal{A})u_T = -\widetilde{T}$. (This can be verified by direct computation.) In fact, this PDE (with appropriate boundary condition) can be used to *define* u_T:

Theorem. u_T is the unique solution to the following problem:

$$\begin{cases} (X + \mathcal{A})u = -\widetilde{T} & \text{on } SM \\ \partial_- u \equiv 0 \end{cases}$$

- proven by applying the method of characteristics. The characteristics in this case are the curves $\widetilde{\gamma_\theta}(t) = \phi_t^X(\theta)$. There are some details, but the proof isn't relevant to the thread we're following.

1.2 Two and a half Explicit Examples

1.2.1 Euclidean Disc

We'll compute the (unattenuated) ray transforms of some basic tensors for $M = \overline{\mathbb{D}}$. In this example, $\partial M = S^1$, geodesics are straight lines, and τ is as follows:

$$\begin{aligned}\tau(\theta) &= -(x \cdot \underline{v}) + \sqrt{(x \cdot \underline{v})^2 - \|x\|^2 + 1} \\ &(= -2(x \cdot \underline{v}) \text{ on } \partial SM)\end{aligned}$$

The ray transform of a tensor $T = T_I \, dx^I$, where the I is a multi-index, is:

$$\begin{aligned}u_T &= \int_0^{\tau(\theta)} T_I(\gamma_\theta(t)) \cdot dx^I(\dot{\gamma}_\theta(t)) \, dt \\ &= \underline{v}^I \int_0^{\tau(\theta)} T_I(x + t\underline{v}) \, dt\end{aligned}$$

Notice how the factor $dx^I(\dot{\gamma}_\theta(t))$ reduces to a constant($\underline{v}^I$) and so doesn't play any part in the integral. This means that the only factor that determines how hard it is to actually evaluate the integral is the coefficient function T_I, and we can write down the transforms of a bunch of simple tensors quite easily. Below are recorded the ray transforms of some specific simple tensors:

dx^I	$\underline{v}^I l_\theta$
$x \, dx^I, \; y \, dx^I$	$\underline{v}^I \left(\frac{1}{2} v_1 l_\theta^2 + x_1 l_\theta\right)$, *simile*
$x^2 \, dx^I, \; y^2 \, dx^I$	$\underline{v}^I \left(\frac{1}{3} v_1^2 l_\theta^3 + x_1 v_1 l_\theta^2 + x_1^2 l_\theta\right)$, *simile*
$xy \, dx^I$	$\underline{v}^I \left(v_1 v_2 l_\theta^3 + \frac{1}{2}[x_1 v_2 + x_2 v_1] l_\theta^2 + x_1 x_2 l_\theta\right)$
$x \, dx + y \, dy$	$\frac{1}{2} l_\theta^2 + (x \cdot \underline{v}) l_\theta$
$x \, dy - y \, dx$	$\det [x \vert \underline{v}] \, l_\theta$
$(x^2 + y^2) \, dx^I$	$\underline{v}^I \left(\frac{1}{3} l_\theta^3 + (x \cdot \underline{v}) l_\theta^2 + \|x\|^2 l_\theta\right)$
$x^2 \, dy^{\otimes 2} - y^2 \, dx^{\otimes 2}$	$v_1 v_2 \det [x \vert \underline{v}] \, l_\theta^2 + (x_1^2 v_1^2 + x_2^2 v_2^2) l_\theta$

1.2.2 Spherical Cap

Slightly more complex $M =$ spherical cap of radius 1 with height $H \in (0,1)$. We have $\partial M =$ circle of radius $\sqrt{1-(1-H)^2}$ in the xy plane, and geodesics are given by:

$$\gamma_\theta(t) = C + \cos(t)\overrightarrow{Cp} + \sin(t)\underline{v}$$

where

- $\theta = (p, \underline{v}) = (p_1, p_2, p_3 | v_1, v_2, v_3)$
- $C = (0, 0, H-1)$, the centre of the sphere

We can then compute the length of geodesics:

$$\tau(\theta) = \arccos\left(\frac{H^c}{\sqrt{v_3^2 + (p_3^+)^2}}\right) + \arctan\left(\frac{p_3^+}{v_3}\right) \;\; (\pm 2k\pi \text{ fudge factor?})$$

where:

- $H^c = 1 - H$
- $p_3^+ = p_3 + H^c$

Finally, we can use this information to compute the (unattenuated) ray transform. Unlike the previous example, the factor $dx^I(\dot{\gamma}_\theta)$ in the integral doesn't reduce to a constant, and so the integrals are a lot harder. Hence we'll only do one specific tensor: the one whose expression in the coordinate system $\varphi \coloneqq \pi_{xy}$ is ${}^\varphi T = dx$. We find:

$$u_{dx}(\theta) = v_1 \sin(\tau_\theta) + p_1 \left[\cos(\tau_\theta) - 1\right]$$

$$= \frac{1}{v_3 + (p_3^+)^2}\left[H^c(p_3^+ v_1 + p_1 v_3) + \left(v_1 v_3 \sqrt{v_3^2 + (p_3^+)^2} - H^c p_1 v_3\right)\sqrt{1 - \frac{(H^c)^2}{v_3^2 + (p_3^+)^2}}\right]$$

The process of expanding the ray transform into coordinates to obtain the preceding expression is quite a nightmare; the following intermediates are recorded here for posterity:

$$\sin(\tau_\theta) = \frac{\sqrt{1-A^2} + AB}{\sqrt{1+B^2}} \qquad A = \frac{H^c}{\sqrt{v_3^2 + (p_3^+)^2}}$$

$$\cos(\tau_\theta) = \frac{A - A\sqrt{1-A^2}}{\sqrt{1+B^2}} \qquad B = \frac{p_3^+}{v_3}$$

1.2.3 Pringle (hyperbolic paraboloid)

This is hardly an example at all, but we'll record some facts nonetheless. Here M is the graph of $x^2 - y^2$ over $\bar{\mathbb{D}}$. It turns out that the geodesics can't even be written out in elementary terms, so it's very dubious that ray transforms could end up being expressible in elementary terms. The geodesic equations are recorded here:

$$\begin{cases} \ddot{x} + \frac{4}{1+4x^2+4y^2}(x\dot{x}^2 - y\dot{y}^2) = 0 \\ \ddot{y} + \frac{4}{1+4x^2+4y^2}(y\dot{y}^2 - x\dot{x}^2) = 0 \end{cases}$$

1.3 The Problem

The question is whether or not the geodesic ray transform of symmetric covariant tensors is injective. That is, can a function/tensor be recovered from its integrals over geodesics?

The specific type of attenuations we work with are of the following form:

$$\mathcal{A} = \widetilde{\mathbf{A}}(\theta) + \mathbf{\Phi}(x)$$

- $\mathbf{A} \in \Omega^1(M)^{n\times n}$ is a matrix of 1-forms. This is called a **general connection**. The lifting is applied entry-wise.
- $\mathbf{\Phi}$ is just a unitary matrix-valued function on M. This is called a **Higgs field**.

The type of attenuation we consider is chosen smartly so that we can interpret it as the connection form of a connection on the bundle in which the tensor takes values, and then it miraculously disappears into the notation when we pass to the PDE formulation of the problem (see 1.4.3)

1.4 The Solution

1.4.1 Natural Obstruction to Injectivity

First, we need to clarify what is meant by "injectivity" in this context. When we're talking about the ray transform of functions (i.e. 0-tensors), injectivity has the usual meaning. When we're talking about tensors of positive order, true injectivity is never possible: all (symmetric covariant) tensors have a (unique) "exact" component (called the potential part, see C.2.3) that vanishes under the ray transform (recorded in the proposition below), so any two tensors with the same potential part will naturally be indistinguishable based on only their ray transforms. Therefore, the most we can hope for is that we would still be able to distinguish the remainders (called the solenoidal components). This is referred to as **s-injectivity** of the ray transform. So "injectivity" in the case of positive-order tensors really means s-injectivity.

Proposition. All potential tensors vanish under the ray transform.

Proof. Potential tensors have the form dS, where $S|_{\partial M} \equiv 0$. We want to show that $I[dS] = 0$. Consider this: the inner derivative is essentially differentiation along geodesics, while the ray transform is integration along geodesics. Hence $I[dS]$ can be evaluated by restricting to geodesics and then applying the fundamental theorem of calculus. Due to the boundary condition on S, the difference in boundary values is 0, completing the argument. ■

1.4.2 Theorem and Proof Outline

As discussed in the previous section, the best we can hope for is s-injectivity. The work in [PS18] proves the following:

Theorem. The attenuated ray transform is s-injective for (symmetric covariant) tensors of all orders on (compact dissipative) manifolds of negative curvature.

The proof is best thought of in two parts: the proof of the Carleman estimate, and then the proof of the theorem itself using the Carleman estimate. The proof of the Carleman estimate is the content of Chapter 4; here we'll outline the proof of the theorem itself. The two flowcharts in Appendix A.2 demonstrate the structure of these two parts graphically.

At this point, the reader is directed to read Appendices B.1, B.2, B.3 before continuing.

Rather than working with the property of s-injectivity as it's defined, we'll reformulate it in the language of PDE's:

Theorem (Reformulation). The ray transform on M is s-injective $\iff$ for all polynomials f, we have $\partial_+ u_f \equiv 0 \implies u_f$ is polynomial.

Proof: next section (1.4.3)

This opens the doors to all the tools from PDE theory. Our starting point is the Carleman estimate for the geodesic vector field:

Theorem 1.1/6.2 (The Carleman Estimate). On a compact Riemannian manifold with negative sectional curvature $\leq -\kappa < 0$ for some $\kappa > 0$ (compact $\implies$ bounded away from 0), the following holds for any $\tau \geq 1$ and $m \in \mathbb{Z}_+$:

$$\sum_{l=m}^{\infty} e^{2\tau\varphi_l} \|u_l\|^2 \leq \frac{(d+4)^2}{\kappa\tau} \sum_{l=m+1}^{\infty} e^{2\tau\varphi_l} \|(Xu)_l\|^2$$

where:

- $u \in C^\infty(SM)$, with $u|_{\partial SM} \equiv 0$ if M has boundary
- X is the geodesic vector field (acting as a differential operator)
- l as subscript refers to decomp w.r.t. vertical spherical harmonics
- $\|\cdot\|$ is the norm on $L^2(SM)$
- $\varphi_l = \log(l)$
- $d = \dim(M)$

Proof: Ch. 4

Using the Carleman estimate, we prove the following general theorem about (possibly nonlinearly) attenuated transport equations:

Theorem 1.4/9.1.

(i) Under the same hypotheses as Theorem 1.1/6.2, it's known that all smooth solutions to the attenuated transport equation $Xu + \mathcal{A}(u) = -f$ (with boundary condition $\partial u \equiv 0$) are polynomial when:
   - f itself is polynomial
   - $\mathcal{A}$ is any operator on $C^\infty(SM, \mathbb{C}^n)$ that satisfies $\|(\mathcal{A}(u))_l\| \leq R \cdot (\|u_{l-l}\| + \|u_l\| + \|u_{l+1}\|)$ for $l \geq$ some $l_0 \geq 2$

(ii) $\deg(f) \leq \max\{l_0 - 1, \deg(f), 2C_{d,\kappa}R\} - 1$
   - $C_{d,\kappa}$ is defined through the course of the proof

Proof: 2.1

We then piggy-back off that to prove our desired result, which we state as Theorem 1.5/9.2:

Theorem 1.5/9.2 (Main theorem of PS18).

(i) Under the following conditions, $u_f^{A+\Phi}$ is a polynomial of degree $\leq \deg(f) - 1$:
   - M satisfies the hypotheses of Theorem 1.1/6.2 and is non-trapping
   - either ∂M is strictly convex or $\text{supp}(f) \subset SM^\circ$
   - f is polynomial
   - $f \in \ker I^{A+\Phi}$

(ii) In the case that f is homogeneous (such as when $f = \widetilde{T}$), u_f is also homogeneous, and its degree is exactly $\deg(f) - 1$

Proof: 2.2

Remarks

- Whereas the first Carleman estimate applies to manifolds of strictly negative curvature, the second Carleman estimate (Theorem 1.2) applies to manifolds with nonpositive curvature – under the addition assumption that they're also simple or Anosov. While they remark (bottom of p.5) that it can be used to obtain s-injectivity for this class of manifolds in a manner analogous to that for manifolds of strictly negative curvature, they note there that two extra ingredients are needed; one of which would restrict the applicability to attenuations consisting of only a Higgs field (i.e. no general connection), and the other is an "additional regularity condition" for which they don't have a proof.
- τ is irrelevant for the purposes of the unattenuated transport equation. It comes into play when we consider attenuations. See bottom of p 4/top of p. 5 in [PS18]

1.4.3 Reformulation as PDE problem

The reformulation relies on the following facts:

1. The lifting/evaluation operation $\widetilde{\cdot}$ is a bijection from symmetric (covariant) tensors on M to homogeneous polynomials on $\partial_+ SM$.
2. (On SM,) being polynomial is equivalent to being finite-degree and smooth
3. $\deg\left(\widetilde{T}\right) = \deg(T)$
4. u_T solves the system
$$\begin{cases} Xu = -\widetilde{T} \\ \partial_+ u = I\left[T\right] \\ \partial_- u \equiv 0 \end{cases}$$
5. $\widetilde{dT} = X\widetilde{T}$
6. $(Xf)^{\text{tens}} = d(f^{\text{tens}})$ **if** f is polynomial (otherwise it couldn't be tensorized)

The presence of attenuations introduces complexity into the reformulation, so we'll do it in three steps based on increasing level of complexity.

Unattenuated ($\mathcal{A} = 0$)

For some reason, I find it clearest to do this purely symbolically. We begin with the statement "I is s-injective on the manifold M":

$$\underbrace{\forall T \in \text{Sym}^m(M)}_{\varphi}\left[I[T] \equiv 0 \implies \exists S \in \text{Sym}^{m-1}(M) \text{ s.t. } T = dS \text{ and } S|_{\partial M} \equiv 0\right]$$

Since $\widetilde{(\cdot)}$ is a bijection $\text{Sym}^m \to \mathscr{P}_m$, we can "reparametrize". We replace φ with "$\forall f \in \mathscr{P}_m$", and then replace each instance of T with f^{tens}. We do the same for S, replacing it by h. After both changes:

$$\forall f \in \mathscr{P}_m\left[\underbrace{I[f^{\text{tens}}]}_{=\partial_+ u_f} \equiv 0 \implies \exists h \in \mathscr{P}_{m-1} \text{ s.t. } \underbrace{f^{\text{tens}} = d(h^{\text{tens}})}_{\iff f = Xh} \text{ and } \underbrace{h^{\text{tens}}|_{\partial M} \equiv 0}_{\iff \partial h \equiv 0}\right]$$

Without all the clutter, it reads:

$$\partial_+ u_f \equiv 0 \implies f = Xh \text{ for some polynomial } h \text{ with } \partial h \equiv 0$$

Since the transport equation has a unique solution for that boundary condition (see C.3.1), h can only be u_f. So really, what the above statement says is:

$$\partial_+ u_f \equiv 0 \implies u_f \text{ is a polynomial}$$

Attenuated with general connection only ($\mathcal{A} = \mathbf{A}$)

This case can be reduced to the previous case easily: look at u as taking values in the trivial bundle $SM \times \mathbb{C}$ instead of $\mathbb{C}$ proper. If we give the bundle a different connection, then we get different exterior and inner derivatives. The connection gives a connection form A, and the connection form back-determines the connection, so we denote the corresponding exterior/inner derivatives by d^A. In the same way that $d = X$, we also have $d^A = X + A$. Then what we're trying to show becomes $I[T] = 0 \implies T = d^A S$, and it works the same way.

Attenuation including Higgs field ($\mathcal{A} = \mathbf{A} + \mathbf{\Phi}$)

This case is more little interesting, because now the LHS of the transport equation is no longer homogeneous: Xu and Au are $\deg(u) + 1$, but Φu is still only $\deg(u)$. Thm 4.6 of [GPSU16] shows how to do this.

Chapter 2

Proofs of Main Theorems

2.1 Proof of Theorem 1.4/9.1

Outline

1. Estimate $\|(Xu)_l\|^2$in terms of certain $\|u_l\|$'s (only holds for l sufficiently large).
2. Chain the estimate from Step 1 with the Carleman estimate 1.1/6.2 and do some black magic to get an inequality $\sum_m^\infty \omega_l \|u_l\|^2 \leq K \sum_m^\infty \omega_l \|u_l\|^2$. (The ω_l's are weights that come out in the proof.)
3. Absorb RHS into LHS (requires knowledge of the constant K) to get a weighted sum bounded above by 0.
4. Conclude that u has finite degree. We also get an explicit bound on the degree.

Proof

Step 1:
First, decompose the transport equation into individual frequencies:

$$(Xu)_l + \mathcal{A}(u)_l = -f_l$$

Notice that for $l \geq \deg(f)+1$, the RHS $= 0$. Hence for such l we have $\|(Xu)_l\| = \|\mathcal{A}(u)_l\|$. For $l \geq l_0$, we can apply the bound in the statement of the theorem. Hence $l \geq \max\{\deg(f)1, l_0\}$ gives:

$$\|(Xu)_l\| \leq R \cdot (\|u_{l-1}\| + \|u_l\| + \|u_{l+1}\|)$$

To convert this into a version where all terms are squared (which is necessary in order for the estimate to be compatible with the Carleman estimate), square the RHS and use the fact that $2ab \leq a^2 + b^2$ to estimate away the cross-terms:

$$\|(Xu)_l\|^2 \leq 3R^2 \cdot \left(\|u_{l-1}\|^2 + \|u_l\|^2 + \|u_{l+1}\|^2\right)$$

Step 2:
If we take $m \geq \max\{\deg(f), l_0 - 1\}$ in the Carleman Estimate 1.1/6.2, then every term on

its RHS can be bounded with the estimate from Step 1 (since the frequencies involved are $l \geq m+1$ = exactly the values of l for which the bound from Step 1 holds). We get:

$$\sum_{l=m}^{\infty} l^{2\tau}\|u_l\|^2 \leq \frac{3CR^2}{\tau}\sum_{l=m+1}^{\infty} l^{2\tau}\left(\|u_{l-1}\|^2 + \|u_l\|^2 + \|u_{l+1}\|^2\right)$$

As in 1.1/6.2, this holds for any $\tau \geq 1$. Reordering the terms, adding a bit extra to the frequency-$(m+1)$ and -$(m+2)$ terms (for symmetry), and re-indexing:

$$\leq \frac{3CR^2}{\tau}\sum_{l=m}^{\infty} \underbrace{\left[(l-1)^{2\tau} + l^{2\tau} + (l+1)^{2\tau}\right]}_{\leq 2(l+1)^{2\tau} \text{ for all } l \geq 1} \cdot \|u_l\|^2$$

Using the fact that $(l+1)^{2\tau} \leq e l^{2\tau}$ for $l \geq 2\tau$, we can write (if we additionally assume that $m \geq 2\tau$):

$$\leq \frac{6eCR^2}{\tau}\sum_{l=m}^{\infty} l^{2\tau}\|u_l\|^2$$

<u>Step 3:</u>
Now we want to absorb the RHS into the LHS by choosing τ. Any $\tau = (1+\varepsilon)6eCR^2$ will work.

<u>Step 4:</u>
The above choice of τ allows us to conclude that u has finite degree $\leq m-1 = \max\{\deg(f), l_0 - 1, 2(1+\varepsilon)6eCR^2\} - 1$.

■

Remarks

1. Regarding Step 3: In [PS18], their τ corresponds to $\varepsilon = 1$. While the choice of ε makes no difference towards concluding that u has finite degree, it does affect the *bound* on the degree, which I've recorded in Step 4.
2. For the purpose of this report, the bound on the degree is actually irrelevant. We only use the fact that the degree *is* finite, which goes into the proof of 1.5/9.2.

2.2 Proof of Theorem 1.5/9.2

Outline

The idea is to start with a solution to a simpler problem, and then upgrade it to a solution of the present problem.

1. Existence for simpler problem
2. Upgrade the boundary condition
3. Smoothness
4. Finitude of degree (by appealing to 1.4/9.1)
5. Refinement of Step 4: bounding the degree (by contradiction); uses GPSU16-5.2

Proof

Step 1:
We begin with a solution of the simpler problem

$$\begin{cases} f = -(X + A + \Phi)u & \text{on } SM \\ \partial_- u \equiv 0 \end{cases}$$

(from the problem in the statement of the theorem, we've weakened the boundary condition and discarded the smoothness and degree requirements.)

Step 2:
Under the hypothesis that $f \in \ker I_{A+\Phi}$, we can actually conclude that the solution to the problem in Step 1 satisfies the stronger boundary condition $u \equiv 0$ on *all* of ∂SM.

Step 3:
With the stronger boundary condition from step 2, we can appeal to existing regularity results to obtain smoothness. The two cases in the third hypothesis are treated separately. We state the results here, but otherwise take them as black boxes.

Case 1: ∂M is strictly convex – Prop. 5.2 of [PSU12] (p.16)

Case 2: $\text{supp}(f) \in SM^\circ$ – Prop. 8.1 of [PS18] (p.39)

Step 4:
Now that we know that f is smooth, we can appeal to 1.4/9.1 (with $\mathcal{A} = A + \Phi$ and $l_0 = \max\{\deg(f) - 1, 2\}$. We conclude that u has finite degree.

Step 5:
Assume, towards a contradiction, that $\deg(u) \not\leq \deg(f) - 1$. Note that the previous inequality implies that $\deg(u) \geq \deg(f)$. By rewriting $X + A = X^A = X_+^A + X_-^A$, the transport equation looks like this:

$$X_+^A u + X_-^A u + \Phi u = -f$$

Now consider the frequency-$(\deg(u)+1)$ component of each side:

$$\underbrace{(X_+^A u)_{\deg(u)+1}}_{=X_+^A(u_{\deg(u)})} + \underbrace{(X_-^A u)_{\deg(u)+1}}_{=X_-^A(u_{\deg(u)+2})} + \underbrace{(\Phi u)_{\deg(u)+1}}_{=\Phi(u_{\deg(u)+1})} = -f_{\deg(u)+1}$$

The second and third terms on the LHS vanish because u doesn't have any frequency-$(\deg(u)+1)$ or -$(\deg(u)+2)$ terms, and the RHS vanishes by the assumption we made. Denoting $u_{\deg(u)}$ – the top-degree term in u – by u^{top}, we're left with $X_+^A(u^{\text{top}}) = 0$.

Now, since $u^{\text{top}}|_{\partial(SM)} \equiv 0$, GPSU16-5.2 implies that $u^{\text{top}} \equiv 0$ – a contradiction (by definition, the top-degree component is nonzero).

We conclude that $\deg(u) \leq \deg(f) - 1$.

∎

Chapter 3

The Pestov Identity

3.1 Generalities

The Pestov identity is a fundamental energy identity concerning the operator $P \coloneqq \overset{\mathrm{v}}{\nabla} X : C^\infty(SM, \mathbb{C}^n) \to \mathcal{Z}^n$ ($\mathcal{Z}$ to be defined shortly). The original/most basic version was supposedly introduced by Pestov and Sharafutdinov in [PS88] (which seems to exist only in print); in [PSU13] it's noted that it has been used in most proofs of injectivity results for the ray transform since work on the problem began (p.7). In the same paper, they record it as (p.9):

$$\|VXu\|^2 = \|XVu\|^2 - (KVu, Vu) + \|Xu\|^2$$

In this instance, V is the vertical vector field, which acts the same as $\overset{\mathrm{v}}{\nabla}$ since there's only one vertical direction in the 2D setting, and K is the usual sectional curvature.

The analogue for higher dimensions was introduced in [PSU15] (Prop 2.2, p.12), and reads as follows:

$$\left\|\overset{\mathrm{v}}{\nabla} Xu\right\|^2 = \left\|X\overset{\mathrm{v}}{\nabla} u\right\|^2 - \left(R\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right) + (d-1)\|Xu\|^2$$

In order to interpret these identities properly, we must note that some normed quantities take values in $\mathbb{C}^n$, while others take values in TM. To nail down the appropriate notion of norm/inner product for the TM-valued quantities, we introduce the function space $\mathcal{Z}$.

At this point, the reader is directed to read Appendix B.4 before continuing.

Throughout the suite of papers leading to [PS18], a series of generalizations have been obtained. The next section covers those presented in [PS18]

3.2 Pestov with Connection

In this paper, they derive three new versions, which are recorded in the lemma below. In addition to generalising it to address general (i.e. not necessarily unitary) connections, they also introduce the notion of expressing it in terms of the decomposition $X_\pm$. Interestingly, none of these results are used full-strength in the proof of the Carleman estimate; we just end up setting $A = 0$, which reduces to a version of the Pestov identity equivalent to the one presented in [PSU15].

Taking a look at the flowchart, let's pay some attention to the step where we set $A = 0$. In all the work up to that point, they record what happens when A is allowed to be fully general. But all that knowledge isn't necessary to prove the Carleman estimate: it all flows through the filter that sets $A = 0$, effectively forgetting that information. Since the main goal of the present report is the proof of the Carleman estimate, we won't provide the proofs of any of their Pestov identities (it's all just symbol-pushing, anyways).

Lemma (Pestov identities from [PS18]):

(i) Lemma 4.2 – A fully general:

$$\left(\overset{\mathrm{v}}{\nabla} X^A u, \overset{\mathrm{v}}{\nabla} X^{-A^*} u\right)_{\mathcal{Z}} = \left(X^A \overset{\mathrm{v}}{\nabla} u, X^{-A^*} \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(\widetilde{R} \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F^A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + (d-1)\left(X^A u, X^{-A^*}\right)_{\mathcal{Z}}$$

(ii) Lemma 4.3 – A fully general (in terms of $X^A_\pm$):

$$\left(X^A_- u, X^{-A^*}_- u\right)_\alpha - \left(\widetilde{R} \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F^A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + (Z_A(u), Z_{-A^*}(u))_{\mathcal{Z}} = \left(X^A_+ u, X^{-A^*}_+ u\right)_\beta$$

(iii) Proposition 4.4 – A unitary (in terms of $X^A_\pm$):

$$\left\|X^A_- u\right\|^2_\alpha - \left(\widetilde{R} \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F^A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + \|Z_A(u)\|^2_{\mathcal{Z}} = \left\|X^A_+ u\right\|^2_\beta$$

(iv) $A = 0$ (in terms of $X_\pm$):

$$\|X_- u\|^2_\alpha - \left(\widetilde{R} \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + \|Z(u)\|^2_{\mathcal{Z}} = \|X_+ u\|^2_\beta$$

where:

- $u \in C^\infty(SM, \mathbb{C}^n)$, with $\partial u \equiv 0$ if M has boundary
- $Z_A(u)$ is the $\overset{\mathrm{v}}{\mathrm{div}}$-free part of $\overset{\mathrm{h}}{\nabla}_A$
- subscripts α, β indicate a weighted version of the standard inner product: $\|u\|^2_\alpha = \sum \alpha_l \|u_l\|^2$, with the following weights:

$$\alpha_l = \begin{cases} d-1 & l = 0 \\ (2l+d-2)\left(1 + \frac{1}{l+d-2}\right) & l \geq 1 \end{cases}$$

$$\beta_l = \begin{cases} 0 & l = 0, 1 \\ (2l+d-2)(1-1/l) & l \geq 2 \end{cases}$$

The following table records how these fit in with other versions from previous papers:

	"default"	in terms of $X_\pm$
no connection	[PSU15]	(taking $A = 0$ in Prop 4.4)
unitary connection	[GPSU16]	Prop 4.4
general connection	Lemma 4.2	Lemma 4.3

3.3 The Single-Frequency Pestov Identity

The single-frequency Pestov identity just records how the unitary Pestov identity of Proposition 4.4 simplifies when you plug in a spherical harmonic (i.e. a function with only one nonzero frequency component): all the terms with the "wrong" degree in the sums defining $\|-\|_\alpha, \|-\|_\beta$ vanish. And those are the only terms that change; the other ones just stay the same:

Proposition 5.1 (Single-Frequency Pestov Identity).

$$\alpha_{l-1} \left\|X_-^A u\right\|^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u, \overset{\mathrm{v}}{\nabla}u\right)_{\mathcal{Z}} - \left(F^A(u), \overset{\mathrm{v}}{\nabla}u\right)_{\mathcal{Z}} + \|Z_A(u)\|_{\mathcal{Z}}^2 = \beta_{l+1} \left\|X_+^A u\right\|^2$$

- M compact; with or without boundary
- A a unitary connection
- $l \geq 0$
- $u \in \Omega_l$
- $u|_{\partial SM} \equiv 0$ if M has boundary
- α_l, β_l as defined in the previous section

3.4 Frequency Localization

The idea here is that we would like to say that the right and left sides of the unitary Pestov identity of Proposition 4.4 can be evaluated by summing each of the single-frequency identities corresponding to the components of the input function.

Lemma 5.2 (Frequency-Localised Pestov Identity). For $u \in C^\infty(SM)$ and unitary connection A we have:

$$
\begin{aligned}
&\left\|X_-^A u\right\|_\alpha^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F_A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + \|Z_A(u)\|_{\mathcal{Z}}^2 \\
&= \sum_{l=0}^{\infty}\left[\alpha_{l-1}\left\|X_-^A u_l\right\|^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla} u_l, \overset{\mathrm{v}}{\nabla} u_l\right)_{\mathcal{Z}} - \left(F_A(u_l), \overset{\mathrm{v}}{\nabla} u_l\right)_{\mathcal{Z}} + \|Z_A(u_l)\|_{\mathcal{Z}}^2\right] \\
&= \sum_{l=0}^{\infty}\beta_{l+1}\left\|X_+^A u_l\right\|^2 \\
&= \left\|X_+^A u\right\|_\beta^2
\end{aligned}
$$

- **Note:** the statement of Lemma 5.2 doesn't actually say what they want it to say; as it's written there, they subtract the RHS's from the LHS's *within* each summand, so that all summands reduce to 0 (as does the RHS). Hence their equation reads $\sum 0 = 0$, which doesn't say anything meaningful. The proper way to write it out is to have a four-way equality as presented here

Public Service Announcement: Major Discrepancy

Multiple times in [PS18], they emphasise that Lemma 5.2 is a key point in the proof of the Carleman estimate:

- abstract: *"the proof is based on showing that the Pestov identity localises in frequency"*
- p.8: *"an important observation of the present paper", "paves the way for the Carleman estimate"*
- p.9: *"key result on frequency localisation"*
- p.21: *"this will be a very important observation in what follows"*

However, I was not able to find any way in which this proposition was used. As demonstrated by the flowcharts in the appendix, I meticulously (in my opinion, at least), traced the dependencies between all the results in the paper, so I'm pretty confident in claiming that it doesn't get used anywhere. I also searched the PDF for "5.2", "localiz" and it didn't turn up any instances of it being invoked by name in a proof.

Proof outline

1. expand u's in the Pestov identity of Proposition 4.4 into components; move sums out of the inner products and norms.

2. notice that there are three types of cross-terms that need to vanish for the localisation to hold
3. prove that they do, in fact, vanish. This step is recorded and proven as Lemma 5.3:

Lemma 5.3. For $u \in C^\infty(SM)$ and unitary connection A we have:

(i) $\left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u_l, \overset{\mathrm{v}}{\nabla}u_m\right)_{\mathcal{Z}} = 0$

(ii) $\left(F^A u_l, \overset{\mathrm{v}}{\nabla}u_m\right)_{\mathcal{Z}} = 0$

(iii) $(Z_A(u_l), Z_A(u_m))_{\mathcal{Z}} = 0$

- Proof of this proposition is over the next three sections
- Only the proof of the first assertion is documented in this report because I typed it up before I realised the major discrepancy discussed above. Since the main goal of this report is to expound the proof of the Carleman estimate, I abandoned the other two assertions once I realised that they may not be relevant

3.4.1 Proof of Lemma 5.3(i)

WTS $\left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u_l, \overset{\mathrm{v}}{\nabla}u_m\right)_{\mathcal{Z}} = 0$ when $l \neq m$

Proof Outline

1. write the inner product as an integral
2. factor into two integrals by Fubini/disintegration (see C.5)
3. rewrite the integrand. Involves properties of spherical harmonics and symmetries of the curvature tensor
4. using the new expression for the integrand (from Step 3), show that the inner integral =0 always. Involves properties of spherical harmonics. Breaks down into two cases.

Proof

Step 1:

$$\left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u_l, \overset{\mathrm{v}}{\nabla}u_m\right)_{\mathcal{Z}} = \int_{SM} \underbrace{\left\langle \widetilde{R}_\theta\overset{\mathrm{v}}{\nabla}u_l, \overset{\mathrm{v}}{\nabla}|_\theta u_m\right\rangle_{N_\theta}}_{=:\ A_x(\underline{v})} \mathrm{d}\mu_L(\theta)$$

(While $A_x(\underline{v})$ can be written as just being a function of θ, we look at the components x and $\underline{v}$ of θ differently, since we don't need to vary x in this context. Accordingly, we permanently drop x from the notation.)

Step 2:

$$= \int_M \underbrace{\int_{S_xM} A_x(\underline{v})\,\mathrm{d}\mu_{S_xM}(\underline{v})}_{=:\ I(x)} \mathrm{d}\mu_M(x)$$

To show that $I(x) = 0\ \forall x \in M$, we need to substantially rewrite $A(\underline{v})$. We will also suppress $\underline{v}$ from the notation for the subsequent computation (to reduce clutter), but it will come back later (whereas x won't). (As an aside, note that each term in the subsequent expansion of A is itself also a function of $\underline{v}$ - and x in the background, but we're permanently ignoring x as discussed previously.)

Step 3:

$$\begin{aligned}
A =& \left\langle R_x\left[\overset{\mathrm{v}}{\nabla}|_\theta u_l, \underline{v}\right](\underline{v}), \overset{\mathrm{v}}{\nabla}|_\theta u_m\right\rangle_{T_xM} \\
=& R_{abcd}\cdot\partial^a[u_l]\cdot v^b\cdot v^c\cdot\partial^d[u_m] \\
=& R_{abcd}\cdot(-mv^a u_l + h^a)\cdot v^b\cdot v^c\cdot(-lv^d u_m + q^d) \\
=& \left[mlu_l u_m\cdot\underbrace{R_{abcd}v^a v^b v^c v^d}_{=:\ B}\right] - \left[mu_l\cdot\underbrace{R_{abcd}v^a v^b v^c q^d}_{=:\ C}\right] - \left[lu_m\cdot\underbrace{R_{abcd}h^a v^b v^c v^d}_{=:\ D}\right] + R_{abcd}h^a v^b v^c q^d
\end{aligned}$$

Note that, like A, each of B, C, D is also a function of $\underline{v}$ (and x). Now we show that each of these three terms $= 0$. Remember that there are sums hiding behind the strings of symbols; cancellation among terms in each sum is what reduces it to 0. We invoke the symmetries of the Riemann curvature tensor R (see Thm in C.2.6).

$$B = \left(\sum_{abcd} \right) R_{abcd} v^a v^b v^c v^d$$

We'll rearrange the terms in the sum according to their first two indices:

- Those whose first two indices are distinct can be grouped in a special way: since $v^a v^b v^c v^d = v^b v^a v^c v^d$, we can write $R_{abcd} v^a v^b v^c v^d + R_{bacd} v^b v^a v^c v^d$ as $(R_{abcd} + R_{bacd})\, v^a v^b v^c v^d$, which will allow us to capitalize on one of the symmetries of R
- Those whose first two indices are identical can be written $R_{iicd} v^i v^i v^c v^d$; these don't occur in pairs, but another symmetry of R takes care of them

In summary:

$$\begin{aligned} &= \sum_{icd} \underbrace{R_{iicd}}_{=\,0} v^i v^i v^c v^d + \sum_{a \neq b} \underbrace{(R_{abcd} + R_{bacd})}_{=\,0} v^a v^b v^c v^d \\ &= 0 \end{aligned}$$

(Note that we could have just as easily rearranged according to the last two indices, since they have the same symmetries as the first two.)

$$\begin{aligned} C &= \left(\sum_{abcd} \right) R_{abcd} v^a v^b v^c q^d \\ &= \sum_{id} \underbrace{R_{iiid}}_{=\,0} v^i v^i v^i q^d + \sum_{a,b,c \text{ not all equal}} \underbrace{(R_{abcd} + R_{bcad} + R_{cabd})}_{=\,0} v^a v^b v^c q^d \\ &= 0 \end{aligned}$$

D - The proof is analogous to that of $C = 0$, except that we rearrange according to the *last* three indices instead of the first

Hence we conclude that $A = R_{abcd} h^a v^b v^c q^d$.

Step 4:

We use this new expression to prove that $I(x) = 0 \ \forall x \in M$. First of all, note that R_{abcd} is a function of only x (being a component of a tensor field), so we can bring it out of the inner integral. Remember, though, that there's a sum hiding in the background, so we need to make sure that it's legal to do so (it is, by linearity of the integral)

$$I(x) = \int_{S_xM} \left[\left(\sum_{abcd} \right) R_{abcd}(x) \cdot h^a v^b v^c q^d \right] \mathrm{d}\mu_{S_xM}(\underline{v})$$

$$= \sum_{abcd} \left[R_{abcd}(x) \underbrace{\int_{S_xM} h^a v^b v^c q^d \, \mathrm{d}\mu_{S_xM}(\underline{v})}_{=\left(h^a v^b, v^c q^d\right)_{L^2(S_xM)}} \right]$$

We'll now show that each of the inner products $\left(h^a v^b, v^c q^d\right)_{L^2(S_xM)} = 0$ when $l \neq m$, which reduces everything to 0 and completes the proof. There are a few cases to consider, based on the values of l and m. Recall that h^a and q^d came out of applications of Lemma 5.4(i). Applying Lemma 5.4(ii), we can expand the inner product as follows:

$$\left(h^a v^b, v^c q^d\right)_{L^2(S_xM)} = (\underbrace{h^{ab}}_{\in\Omega_m}, \underbrace{q^{cd}}_{\in\Omega_l}) + (\underbrace{h^{ab}}_{\in\Omega_m}, \underbrace{f^{cd}}_{\in\Omega_{l-2}}) + (\underbrace{f^{ab}}_{\in\Omega_{m-2}}, \underbrace{q^{cd}}_{\in\Omega_l}) + (\underbrace{f^{ab}}_{\in\Omega_{m-2}}, \underbrace{f^{cd}}_{\in\Omega_{l-2}})$$

(all inner products on the RHS are also in $L^2(S_xM)$)

Recall that the Ω spaces are orthogonal. Hence in the case $l \neq m$ (which is a hypothesis of the theorem, since we don't care what happens when $l = m$), the first and fourth terms vanish. The LHS can only be nonzero (which is what we want to show *can't* happen) if at least one of the remaining two terms on the RHS is nonzero. Due to the orthogonality, each of them term is nonzero in at most one case:

- second term: $l = m + 2$
- third term: $l = m - 2$

We'll now show that they both $= 0$ even in the aforementioned special cases.

Case 1: $l = m + 2$

In this case,

$$I(x) = \sum_{abcd} \left[R_{abcd}(x) \int_{S_xM} h^{ab} f^{cd} \, \mathrm{d}\mu_{S_xM}(\underline{v}) \right]$$

$$= \int_{S_xM} \left[\sum_{abcd} R_{abcd} h^{ab} f^{cd} \right] \mathrm{d}\mu_{S_xM}(\underline{v})$$

Applying an argument similar to the one used in Step 3 to show that $B = 0$ allows us to conclude that the integrand $= 0$ (Previously we appealed to the obvious symmetry of $v^a v^b v^c q^d$ in its first two indices; here we appeal to the symmetry of h^{ab} in its two indices, as given in Lemma 5.4(iii)

Case 2: $l = m - 2$
This case is analogous to case 1.

$\blacksquare$

3.4.2 Proof of Lemma 5.3(ii)

Omitted

3.4.3 Proof of Lemma 5.3(iii)

Omitted

3.5 Commutator Identities

Lemma 4.1 (Commutator identities).

(i) $[X^A, \overset{\text{v}}{\nabla}] = -\overset{\text{h}}{\nabla}_A$

(ii) $[X^A, \overset{\text{h}}{\nabla}_A] = \widetilde{R}\overset{\text{v}}{\nabla} + F^A$

(iii) $\overset{\text{h}}{\text{div}}_A\overset{\text{v}}{\nabla} - \overset{\text{v}}{\text{div}}\overset{\text{h}}{\nabla}_A = (d-1)X^A$

(iv) $[X^A, \overset{\text{v}}{\text{div}}] = -\overset{\text{h}}{\text{div}}_A$

The first three are on $C^\infty(SM, \mathbb{C}^n)$, the last is on $\mathcal{Z}^n$

- **Note:** there's a bit of subtlety masked by the notation. For example, consider the expanded version of the commutator in (i): $X^A(\overset{\text{v}}{\nabla}u) - \overset{\text{v}}{\nabla}(X^A u)$. Notice that the first X^A refers to the action of the geodesic vector field on $\mathcal{Z}^n$, whereas the second is referring to the action on $C^\infty(SM, \mathbb{C}^n)$. So these are like symbolic equalities rather than formal identities within an operator algebra (since each instance of X^A lives in a different algebra)

Proof Outline

1. Express $X, \overset{\text{v}}{\nabla}, \overset{\text{h}}{\nabla}$ in coordinates

$$Xu = v^j \cdot \delta_j u$$
$$\overset{\text{v}}{\nabla}u = \left(\partial^k u\right) \cdot \partial_{x_k}$$
$$\overset{\text{h}}{\nabla}u = \left[\delta^j u - \left(v^k \cdot \delta_k u\right) \cdot v^j\right] \cdot \partial_{x_j}$$

2. Brackets of δ's and ∂'s

$$\left[\delta_{x_j}, \delta_{x_k}\right] = -\left(\sum_{m,l}\right) R^m_{jkl} \cdot y^l \cdot \partial_{y_m}$$

$$\left[\delta_{x_j}, \partial_{y_k}\right] = \left(\sum_{l}\right) \Gamma^l_{jk} \cdot \partial_{y_l}$$

$$\left[\partial_{y_j}, \partial_{y_k}\right] = 0$$

$$((\left[\partial_{x_j}, \partial_{x_k}\right] = 0))$$

$$((\left[\partial_{x_j}, \partial_{y_k}\right] = 0))$$

$$[\delta_j, \delta_k] = -\left(\sum_{m,l}\right) R_{jklm} \cdot v^l \cdot \partial^m$$

$$[\delta_j, \partial_k] = \left(\sum_{l}\right) \Gamma^l_{jk} \cdot \partial_l$$

$$[\partial_j, \partial_k] = v_j \cdot \partial_k - v_k \cdot \partial_j$$

$$\left[\partial_j, v^k\right] = \partial^k_j - v_j \cdot v^k$$

$$\left[\delta_j, v^k\right] = -\left(\sum_{l}\right) \Gamma^k_{jl} \cdot v^l$$

$$\left[\delta_j, \delta^k\right] = \left(\sum_{l}\right) \left[-g^{kl}\left(\sum_{pq} R_{jlpq} \cdot v^p \cdot \partial^q\right) + \left(\partial_{x_j} g^{kl} \cdot \delta_l\right)\right]$$

$$\left[\delta_j, \partial^k\right] = -\left(\sum_{l}\right) \Gamma^k_{jl} \cdot \partial^l$$

3. Adjoints of $X, \overset{\mathrm{v}}{\nabla}, \overset{\mathrm{h}}{\nabla}$ on $\mathcal{Z}$ - this is only used in proving the last identity

$$X^* = -X$$

$$\left(\overset{\mathrm{v}}{\nabla}\right)^* = -\overset{\mathrm{v}}{\mathrm{div}} = \sum_{j} \partial_j$$

$$\left(\overset{\mathrm{h}}{\nabla}\right)^* = -\overset{\mathrm{h}}{\mathrm{div}} = \sum_{j} \delta_j + \Gamma_j$$

4. Prove commutator identities

Chapter 4

The Carleman Estimate

Comparison of estimates

Both estimates have the same form:

$$\sum_{m}^{\infty} e^{2\tau\varphi_l} \|u_l\|^2 \leq C \cdot \sum_{m+1}^{\infty} e^{2\tau\varphi_l} \|(Xu)_l\|^2$$

The table below compares hypotheses and conclusions.

	Case 1 (Theorem 1.1/6.2)	Case 2 (Theorem 1.2)
curvature	negative + bdd away from 0 (i.e. $\leq -\kappa < 0$)	nonpositive (i.e. ≤ 0)
req's on M	compact	simple or Anosov (both $\implies$ compact)
φ_l	$\log(l)$	l (i.e. linear)
τ, m	can both take any vals ≥ 1	can only take sufficiently large vals (thresholds are independent)
C	$\dfrac{(\dim M + 4)^2}{\kappa\tau}$	$\dfrac{24}{\kappa e^{2\tau}}$
κ	curvature bound	? (defined on p.33)
req's on u	$C^\infty(SM)$; $u\|_{\partial SM} \equiv 0$ if M has bdy	$C^\infty_F(SM)$ (= polynomial); $u\|_{\partial SM} \equiv 0$ if M has bdy

4.1 Proof Outline

The second flowchart in Appendix A.2 demonstrates the structure of this part.

The Carleman estimate is obtained as a corollary – see the next section – to the following "re-weighted Pestov identity":

Theorem 6.1. Under the same conditions as Theorem 1.1/6.2, the following holds for $s > {}^{-1}/2$ and $m \in \mathbb{Z}_+$:

$$\sum_{l=m}^{\infty} \left\{ \begin{array}{cc} 2l^{2s+1} & l = m, m+1 \\ (2s+1)(l-1)^{2s} & l \geq m+2 \end{array} \right\} \|X_- u_l\|^2 + \kappa \sum_{l=m}^{\infty} l^{2s+2} \|u_l\|^2 + \sum_{l=m}^{\infty} l^{2s} \|Z(u_l)\|_{\mathcal{Z}}^2$$
$$\leq C \sum_{l=m+1}^{\infty} l^{2s+2} \|(Xu)_l\|^2$$

- See the discussion after the proof for values of the constant C

Proof: 4.4

The inequality in Theorem 6.1 is assembled as follows: from Proposition 5.1 (Single-frequency Pestov identity), we pull out a collection of localized estimates:

Lemma 6.4 ("Localised Estimates"). Under the same conditions as Theorem 1.1/6.2, the following holds for $u \in \Omega_l$:

$$\alpha_{l-1} \|X_- u\|^2 + \kappa \lambda_l \|u\|^2 + \|Z(u)\|_{\mathcal{Z}}^2 \leq \beta_{l+1} \|X_+ u\|^2$$

Proof: 4.5

We then weight each one of the localized estimates (by the sequence $\{\gamma_l\}$) and add them all up. This gives us the estimate of Proposition 6.6:

Proposition 6.6 (Ansatz for Carleman estimate). Under the same conditions as Theorem 1.1/6.2, the following holds for functions of *finite degree*:

$$\sum_{l=m}^{\infty} \left\{ \begin{array}{cc} \alpha_{l-1}\gamma_l^2 & l = m, m+1 \\ (1-\delta_{l-1})(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2) & l \geq m+2 \end{array} \right\} \|X_- u_l\|^2 + \kappa \sum_{l=m}^{\infty} \lambda_l \gamma_l^2 \|u_l\|^2 + \sum_{l=m}^{\infty} \gamma_l^2 \|Z(U)\|_{\mathcal{Z}}^2$$
$$\leq \sum_{l=m+1}^{\infty} \left(1 + \frac{1-\delta_l}{\delta_l} \cdot \frac{\beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2}\right) \frac{\alpha_l \gamma_{l+1}^2 \beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2} \|(Xu)_l\|^2$$

- m is any integer ≥ 1
- $\{\gamma_l\}$ is any sequence in $\mathbb{R}_+$ that satisfies $\alpha_l \gamma_{l+1}^2 > \beta_l \gamma_{l-1}^2$ for $l \geq m+1$
- $\{\delta_l\}$ is *any* sequence in $(0, 1]$
- u is a polynomial (smooth and finite degree)
- $u|_{\partial SM} \equiv 0$ if M has boundary

Proof: 4.6

Remark: don't think of Proposition 6.6 as a *generalization* of Theorem 6.1 (which sort of implies that Theorem 6.1 is proven first), but rather as a first attempt at assembling Theorem 6.1 – an ansatz. What appears prima facie as complications (the gobbledygook of weights) is actually baked-in wiggle room. The philosophy here is: "let's put something together that has the same structure as our desired inequality, but leave ourselves a bunch of wiggle room (in the form of the choice of gammas and deltas), and then figure out how to chose gamma and delta to get us all the way there. From this point, there are two things left to figure out: we need to choose the weights γ_l and δ_l, and then we also need to address the convergence of the series in the case where u is not finite-degree.

4.2 Proof of the Carleman Estimate from Theorem 6.1

Start with the inequality from Theorem 6.1; replace s by $\tau - 1$:

$$\sum_{l=m}^{\infty} \left\{ \begin{array}{cc} 2l^{2\tau-1} & l = m, m+1 \\ (2\tau-1)(l-1)^{2\tau-2} & l \geq m+2 \end{array} \right\} \|X_{-}u_l\|^2 + \underbrace{\kappa \sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2}_{\text{desired LHS}} + \sum_{l=m}^{\infty} l^{2\tau-2} \|Z(u_l)\|_{\mathcal{Z}}^2$$

$$\leq \underbrace{C \sum_{l=m+1}^{\infty} l^{2\tau} \|(Xu)_l\|^2}_{\text{desired RHS (up to constant)}}$$

Notice: the third term on the LHS is always positive regardless of τ, so it can dropped without affecting the validity of the inequality. Moreover, recall that the above inequality comes with the stipulation that $s > -1/2$. That's equivalent to $\tau > 1/2$, which guarantees that $2\tau - 1 > 0$, and so the first term is positive as well and it can also be dropped:

$$\kappa \sum_{l=m}^{\infty} l^{2\tau} \|u_l\|^2 \leq C \sum_{l=m+1}^{\infty} l^{2\tau} \|(Xu)_l\|^2$$

Finally, we'll address the constant. As noted after the proof of Theorem 6.1, we don't have a single value of C that works for all combinations of d and s. Case 2(b) is the "annoying/problematic" one; the others yield a "clean" value for C. It turns out that we can sidestep the problematic case altogether: in our application of the Carleman estimate (in the proof of Theorem 1.4/9.1), we only ever take $\tau \geq 1$ ($\iff$ $s \geq 0$), so we can forget that the present inequality also holds (with a different constant) for $s \in (-1/2, 0)$ without consequence. Under this new circumstance, the value for the constant is $\frac{(d+4)^2}{2\tau-1}$ (unconditionally). Also, we'll replace the denominator by just τ for simplicity. (Doing so is legal because – still under the restriction that $\tau \geq 1$ – it only enlarges the constant.)

Finally, dividing the κ back onto the RHS and replacing the $l^{2\tau}$'s gives the Carleman estimate as stated in Theorem 1.1/6.2:

$$\sum_{l=m}^{\infty} e^{2\tau \log(l)} \|u_l\|^2 \leq \frac{(d+4)^2}{\kappa\tau} \sum_{l=m+1}^{\infty} e^{2\tau \log(l)} \|(Xu)_l\|^2$$

4.3 Choice of Weights

Summary

	Case 1 (Theorem 1.1/6.2)	Case 2 (Thm 1.2)
Case 1	$\gamma_l := l^s$ $\delta_l \equiv {}^1\!/_2$ though even $\delta_l \equiv 1$ would work	
Case 2(a)	ditto	
Case 2(b)	$\gamma_l := \begin{cases} l^s & l \geq l_0 \\ \sqrt{\frac{1}{2}\frac{\alpha_l}{\beta_l}}\gamma_l & l < l_0 \end{cases}$ $\delta_l \equiv {}^1\!/_2$ here $\delta_l \equiv 1$ would *not* work	

*Cases refer to those that arise in the proof of Theorem 6.1

Motivation/Intuition

Now that we have our ansatz Proposition 6.6 for the Carleman estimate, we need to "solve" for the weight sequence $\{\gamma_l\}$. A good place to start our search is by analyzing the case $d = 2$. In this case, the sequences $\{\alpha_l\}, \{\beta_l\}, \{\lambda_l\}$ take on a particularly simple form:

- $\alpha_l = 2l + 2$
- $\beta_l = 2l - 2$
- $\lambda_l = l^2$

Step 1:

The whole point of the weight sequence $\{\gamma_l\}$ is to facilitate the absorption, so any sequence that allows us to do so will work, regardless of what it does to the other terms in the inequality of 6.6. Since they don't factor into the absorption, drop the terms $\sum^\infty \gamma_l^2 \lambda_l \|u_l\|^2$ and $\sum^\infty \gamma_l^2 \|Z(u_l)\|^2$ (this makes the small side even smaller, so it's allowed), take $\delta_l \equiv 1$ (this achieves two effects: first, it greatly simplifies the weights on the RHS, and second, it gives the tightest upper bound, which is the best case scenario for absorption; if it can't be done with $\delta_l \equiv 1$, it can't be done at all, since the part we want to absorb only gets larger). Plugging in the values of the above sequences:

$$\kappa \sum_{l=m}^{\infty} l^2 \gamma_l^2 \|u_l\|^2 \leq 2 \sum_{l=m+1}^{\infty} \frac{(l+1)\gamma_{l+1}^2 (l-1)\gamma_{l-1}^2}{(l+1)\gamma_{l+1}^2 - (l-1)\gamma_{l-1}^2} \|(Xu)_l\|^2$$

Notice that the RHS can be expressed in terms of the simple sequence $r_l := l\gamma_l^2$ (we also express the LHS so that everything is in the same terms):

$$\kappa \sum_{l=m}^{\infty} l r_l \|u_l\|^2 \leq 2 \sum_{l=m+1}^{\infty} \frac{r_{l+1} r_{l-1}}{r_{l+1} - r_{l-1}} \|(Xu)_l\|^2$$

Step 2:

Now we're going to completely absorb the RHS into the LHS. Just like we did in Step 1 of Theorem 1.4/9.1, we're going to bound $\|(Xu)_l\|^2$ in terms of u only:

$$\begin{aligned}\|(Xu)_l\| &= \|(X_+u)_l + (X_-u)_l\| \\ &\leq \|Xu_{l-1}\| + \|Xu_{l+1}\| \\ \|(Xu)_l\|^2 &\leq \|Xu_{l-1}\|^2 + \|Xu_{l+1}\|^2 + 2\|Xu_{l-1}\| \cdot \|Xu_{l+1}\| \\ &\leq 2\left(\|Xu_{l-1}\|^2 + \|Xu_{l+1}\|^2\right)\end{aligned}$$

Since our u will always satisfy $Xu + Au = 0$, we have $\|Xu_l\| \leq \|A\|_\infty \cdot \|u_l\|$. With this:

$$\leq \underbrace{2\|A\|_\infty^2}_{=:R} \cdot \left(\|u_{l-1}\|^2 + \|u_{l+1}\|^2\right)$$

Note: They take $R = 4\|A\|_\infty$. Regardless of whether the constant is 2 or 4, I'm not sure how you can get away with using $\|A\|$ rather than $\|A\|^2$ in the previous step. I tried to compute $\|A\|$, which led me to realize that I have no idea what the "standard" way to norm a matrix of tangent/cotangent vectors is. I couldn't find anything online, and a question that I posted to MathOverflow didn't receive any responses. This particular instance is complicated by the fact that the connection form A doesn't transform tensorially, but I assume that the chosen way to norm it should still be independent of coordinates.

Note 2: I suspect that the choice to use 4 instead of 2 might have something to do with making $R/\kappa > 1$, which is relevant later (see Step 5).

To absorb the RHS into the LHS, split it into two sums and re-index so that all u's have the subscript l. After absorbing, we have:

$$\sum_{l=m}^{\infty} \left\{ \begin{array}{ll} \kappa l r_l - 2R\frac{r_{l+2}r_l}{r_{l+2}-r_l} & l = m, m+1 \\ \kappa l r_l - 2R\left(\frac{r_{l+2}r_l}{r_{l+2}-r_l} + \frac{r_l r_{l-2}}{r_l - r_{l-2}}\right) & l \geq m+2 \end{array} \right\} \cdot \|u_l\|^2 \leq 0$$

In order to conclude that $\|u_l\|^2 = 0\ \forall l$, we need all coefficients to be positive. To make things easier, we can replace the first two coefficients (i.e. those for $l = m$ and $m+1$) with ones having the same form as the rest (i.e. those for $l \geq m+2$); doing so makes them smaller, so if they're still positive after the replacement, then we of course can conclude that the originals are as well. So we want the sequence $\{r_l\}$ to satisfy the following condition:

$$2R\left(\frac{r_{l+2}r_l}{r_{l+2} - r_l} + \frac{r_l r_{l-2}}{r_l - r_{l-2}}\right) \leq \kappa l r_l \qquad \text{for} \quad l \geq m$$

Step 3:
Lets try and pull a simpler inequality out of the above. Notice that if we drop the first summand above, we'll have a relation between r_l and r_{l-2}, which essentially will give us a recursive condition on the growth. (Dropping the second term and working with r_{l+2} and r_l gives the exact same thing after re-indexing.) So dropping the first term and rearranging gives:

$$r_l \geq \left(\frac{2^R/\kappa + l}{l}\right) r_{l-2} \qquad \text{for } \ l \geq m$$

For notational convenience, we let $S := {}^R/\kappa$. By induction, we see that:

$$r_{m+2k} \geq \underbrace{\left[\prod_{j=1}^{k} \frac{m+2j+2S}{m+2j}\right]}_{=:\,P} r_m \qquad \text{for } k \geq 0$$

Step 4:
The growth factor P is quite hard to work with, so we'd like to bound it by something simpler. The simplest bound - I call it the naive bound - is to just replace each factor in the numerator by the smallest and each factor in the denominator by the largest. This turns out to be pretty useless (explained later - see Remark 1 in Step 6). Here's how you might stumble upon a better one: if you pick some a random pairs of (positive) whole numbers to plug in for k and S and write out the whole product explicitly, you notice that the product telescopes when $S < k$: the factors cancel with an offset of S, leaving the greatest S-many factors in the numerator and the smallest S-many factors in the denominator. If we take the naive bound on what's left, we get:

$$P \geq \left(\frac{m+2k+2}{m+2S}\right)^S \geq \underbrace{\left(\frac{m+2k}{m+2S}\right)^S}_{=:\,Q} \qquad \text{for } m \geq 1, k \geq 0$$

(We drop the "+2" from the numerator for simplicity.) While the cancellation argument only works for $S \in \{1, \cdots, k-1\}$, it turns out that the bound holds for any $S \in [1, \infty)$, which can be seen on this interactive graph.

Step 5:
Now we need to investigate the value of ${}^R/\kappa$. That it's positive is trivial, but we also need it to be ≥ 1 so that the bound from the previous step holds. Though I don't know for certain, I have two reasons to suspect that it is, which I'll cover later. So assume that it is.

Step 6:
We have:

$$r_{m+2k} \geq \left(\frac{m+2k}{m+2S}\right)^S r_m \qquad \text{for } m \geq 1, k \geq 0$$

Recall that the $(m+2k)$'s started out as l's. Converting them back, we find:

$$r_l \geq \frac{1}{(m+2S)^S} r_m \cdot l^S \qquad (\text{for } l \geq m \text{ and having same parity as } m)$$

Remark 1: Now we're in a position to see why the naive bound would not have been a good choice in Step 4. It turns out that the naive bound is the same as Q with k and S interchanged; Q puts l in the numerator, but the naive bound would have put it in the denominator - hardly a *growth* condition at all.

Replacing r_l by $l\gamma_l^2$ and rearranging, the corresponding growth condition for γ_l is:

$$\gamma_l \geq \underbrace{\frac{\sqrt{m}\gamma_m}{(m+2S)^{S/2}}}_{c_{m,S}} \cdot l^{\frac{S-1}{2}} \qquad \text{(for } l \geq m \text{ and having same parity as } m\text{)}$$

Hence $\gamma_l \geq c \cdot l^{\frac{R/\kappa - 1}{2}}$

Remark 2: Now I can cite my reasons for believing that $R/\kappa > 1$:

1. in the paper, they conclude that $\gamma_l \geq cl^s$ "for some sufficiently large $s > 0$". Assuming their s refers to $\frac{R/\kappa-1}{2}$, then $s > 0 \iff R/\kappa > 1$.
2. In [GPSU16] the following "related" (in the sense that it's structurally similar to R/κ and both the connection form – which appears in one – and the curvature form – which appears in the other – encode the connection) quantity appears, which here I'll refer to as C:

$$2\left(\frac{\left\|F^{\mathcal{E}}\right\|_{L^\infty}}{\kappa}\right)^2$$

   where $F^{\mathcal{E}}$ is essentially the curvature form. The following appears as a hypothesis in Lemma 4.2 (p.23): "assume m is so large that $\lambda_m \geq C$". Since $\lambda_m = m(m+d-2)$, we have $\lambda_m \geq 1 \forall m$, so the hypothesis would be redundant unless $C > 1$.

Conclusion:

So the point is that the absorption requires $\{\gamma_l\}$ to grow at least polynomially (when talking about growth/complexity classes, "polynomial" allows any positive exponent, including those < 1). On the other hand, they can't grow too quickly or else the terms $\sum^\infty \gamma_l^2 \lambda_l \|u_l\|^2$ and $\sum^\infty \gamma_l^2 \|Z(u_l)\|^2$ won't converge. Luckily, they converge for any polynomial weights (see Step 4 of the proof of 6.1 §**??**)

Let's be clear with what we've done in this section. At the moment, what we've determined is that **if** there is a sequence $\{\gamma_l\}$ that will allow us to do the absorption as we want **in the case** $d = 2$, then that sequence would have to grow polynomially. **However**, we haven't yet shown that any such sequence does indeed work, even for 2 dimensions (let alone higher dimensions); that's the content of 4.4, the next section.

4.4 Proof of Theorem 6.1

We'll introduce some labels:

$$\sum_{l=m}^{\infty}\left\{\begin{array}{ll}\alpha_{l-1}\gamma_l^2 & l=m,m+1\\ (1-\delta_{l-1})\underbrace{(\alpha_{l-1}\gamma_l^2-\beta_{l-1}\gamma_{l-2}^2)}_{=:A} & l\geq m+2\end{array}\right\}\|X_- u_l\|^2+\kappa\sum_{l=m}^{\infty}\lambda_l\gamma_l^2\|u_l\|^2+\sum_{l=m}^{\infty}\gamma_l^2\|Z(u_l)\|_{\mathcal{Z}}^2$$

$$\leq\sum_{l=m+1}^{\infty}\left(1+\frac{1-\delta_l}{\delta_l}\cdot\underbrace{\frac{\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2}}_{=:B}\right)\underbrace{\frac{\alpha_l\gamma_{l+1}^2\beta_l\gamma_{l-1}^2}{\alpha_l\gamma_{l+1}^2-\beta_l\gamma_{l-1}^2}}_{=:C}\|(Xu)_l\|^2$$

Proof outline

1. Show that $A>0$ for $s\geq -1/2$. This does double duty: it shows that our choice of weights satisfies the growth condition, and it helps in pulling out the estimate.
2. Show that $B\leq 3, C\leq \frac{(2+d/2)^2}{(2s+1)}l^{2s+2}$
3. Noting a few other simple estimates and Parting the Red Sea. Up to this point, we have the estimate for functions of finite degree.
4. Justify why the estimate still holds for arbitrary (i.e. not necessarily finite-degree) u

The proof breaks down into three cases, based on the values of d and s:

- Case 1 - $d=2$ (s can be anything)
- Case 2(a) - $d\geq 3, s\geq 0$
- Case 2(b) - $d\geq 3, 0>s>-1/2$

The first two cases are treated simultaneously, since they have the same proof. The last case is quite finicky. Having said that, we can weaken the theorem by forgetting about case 2(b) to no detriment: in using the theorem to prove the main Estimate (Thm 1.1), we assume $s\geq 0$ anyways.

Proof

Step 1:
Notice that the growth estimate is equivalent to $A\geq 0$ Showing this does double duty, because we need an estimate on A to estimate C.
First, rewrite α_l,β_l as follows:

$$\alpha_l=2(l+1)+(d-2)(1-1/l)+\frac{(d-2)^2}{l(l+d-2)}$$
$$\beta_l=2(l-1)+(d-2)(1-1/l)$$

Letting $\psi_p(l):=(l+1)^p-(l-1)^p$, we then have:

$$A=2\psi_{2s+1}(l)+(d-2)(1-1/l)\psi_{2s}(l)+\frac{(d-2)^2}{l(l+d-2)}(l+1)^{2s}$$

Case 1: When $d = 2$, the second and third terms vanish, so that $A = 2\psi_{2s+1}(l)$. Since ψ is exactly the quantity in Lemma 6.8, we conclude that $A \leq 2(2s+1)l^{2s}$ for $s \geq -1/2$ (which corresponds to $p \geq 0$). Since we need strict inequality, we exclude the case $s = -1/2$, and that's where the requirement that $s > -1/2$ in the statement cones from.

Case 2(a): For $d = 3$, none of the terms vanish. The fact that the second term has $p = 2s$ instead of $2s+1$ is why we have to break into two subcases: to estimate this term below, we can only have $s \geq 0$. But both terms reveal themselves to be positive, so the result still holds.

Case 2(b): Omitted

Step 2:
Now we'll bound B and C. We start with B:

$$\begin{aligned} B &= \frac{(l-1)\left(2+\frac{d-2}{l}\right)\cdot(l-1)^{2s}}{(l+1)\left(2+\frac{(d-2)(l-1)}{l(l+1)}\right)\cdot(l+1)^{2s}} \\ &= \frac{2l^2+dl+(d-2)}{2l^2+dl-(2d-2)}\cdot\underbrace{\left(\frac{l-1}{l+1}\right)^{2s+1}}_{\leq 1} \end{aligned}$$

To bound the first factor, consider it as a family of functions of l with d as a parameter. Once you figure out the zeros and asymptotes, you can see that all the functions in the family are decreasing – and bounded – for $l \geq 2$. Plugging in $l = 2$, we get the following family of bounds:

$$\leq 3\cdot\underbrace{\frac{d+2}{d+10}}_{\leq 1}$$

This shows that $B \leq 3$, as desired.

Next we bound C:

$$C = \frac{\alpha_l\beta_l}{2(2s+1)}\cdot\left(\frac{l^2-1}{l}\right)^{2s}$$

For $l \geq 2$, the second factor is bounded above by:

$$\begin{cases} l^{2s} & s \geq 0 \\ (3/4l)^{2s} & s \leq 0 \end{cases}$$

Which gives:

$$\leq \frac{(2l+d)(2l+d-4)}{2(2s+1)}\cdot\underbrace{\max\{1,(3/4)^{2s}\}}_{\leq 2 \text{ for } s > -1/2} l^{2s}$$

Cancel the 2's, drop the -4 (which makes both factors in the numerator the same), and factor out an l from each of them:

$$\leq \frac{(2+d/l)^2}{2s+1} \cdot l^{2s+2}$$

Replacing the l in the numerator by its minimum value of 2 gives the bound.

Step 3:
Now we'll "part the Red Sea". We'll pull our desired inequality out of the inequality of Proposition 6.6 by decreasing all the weights on the smaller (R) side and increasing the weights on the bigger (L) side. The decreases/increases are done using the bounds on A, B, and C from steps 1 and 2 as well as the following two estimates:

$$\alpha_l \geq 2(l+1)$$
$$\lambda_l \geq l^2$$

Here we also fix $\delta_l \equiv 1/2$. The changes are all very straightforward, so they're not detailed here. This completes the derivation of the inequality in the statement of the theorem, but so far we only know it to hold for functions of finite degree.

Step 4:
We now need to show that all the series converge when u is infinite-degree. It's a standard fact that 1-dimensional – and, I think, multi-dimensional – Fourier coefficients of smooth functions decay faster than any polynomial. Now, these results fall under harmonic analysis on toruses, whereas we're working on spheres. I'm assuming they know that the same holds for spheres, though I wasn't able to find anything about it online.

■

The value of the constant

- In cases 1 and 2(a), the value of C obtained is $\frac{(d+4)^2}{2s+1}$
- The value is not optimal, for three reasons: the optimal occurs when $\delta_l \equiv 1$, would also depend on m, and moreover we estimated the value of ours up after fixing δ_l and ignoring m.
- Remark 6.9 (pp.32/33) discusses the optimal constant in these cases. While we don't have an elementary expression for it, they show two things: first, that the optimal constant is asymptotic (in s) to $\frac{1}{2s+1}$, and second that ours is hence at least asymptotic to the optimal. By taking a larger value of s, we can make it as close as necessary to the optimal value

- They don't even bother trying to give an elementary expression for the constant in case 2(b) because it's quite finicky, but they essentially encode the inequality with the optimal value for this case, which is $\frac{\max\{C, 1/2\frac{(d+4)^2}{2s+1}\}}{\min\{c,1\}}$. C – which here represents a different quantity – and c are defined at the top of p.32

4.5 Proof of Lemma 6.4

This is a consequence of 5.1

Outline

1. Set $A = 0$ in the single-frequency Pestov identity 5.1
2. Bound the second term below
3. Rewrite the bound from step 2

Proof

Step 1:
Consider the single-frequency Pestov identity from 5.1:

$$\alpha_{l-1} \left\| X_-^A u \right\|^2 - \left(\widetilde{R} \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u \right)_{\mathcal{Z}} - \left(F_A(u), \overset{\mathrm{v}}{\nabla} u \right)_{\mathcal{Z}} + \| Z_A(u) \|_{\mathcal{Z}}^2 = \beta_{l+1} \left\| X_+^A u \right\|^2$$

When we set $A = 0$, the third term vanishes because $F_0 = 0$ (by definition of F_A). For the other terms, $A = 0$ has the effect of just dropping A from the notation.

Step 2:
To compute the bound, it will help to emphasize that the two arguments of the inner product are functions of θ

$$\begin{aligned} -\left(\widetilde{R}_\theta \overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla}|_\theta u \right)_{\mathcal{Z}} &= -\left(R_x(\overset{\mathrm{v}}{\nabla}|_\theta u, \underline{v})\underline{v}, \overset{\mathrm{v}}{\nabla}|_\theta u \right)_{\mathcal{Z}} \\ &= -\int_M \int_{S_x M} \left\langle \overset{\mathrm{v}}{\nabla}|_\theta u, \underline{v}, \underline{v}, \overset{\mathrm{v}}{\nabla}|_\theta u \right\rangle_{N_\theta} \mathrm{d}\underline{v} \mathrm{d}x \end{aligned}$$

We can absorb the negative in front to switch the third and fourth slots of the integrand. Also, we'll omit the information about the integrals, since we don't have to manipulate them at all ((we're only working with the integrand)). Now we have:

$$= \iint \underbrace{\left\langle \overset{\mathrm{v}}{\nabla}|_\theta u, \underline{v}, \overset{\mathrm{v}}{\nabla}|_\theta u, \underline{v} \right\rangle_{N_\theta}}_{=:I(\theta)}$$

By Prop 4.3.1 in [dC06] (p.94, $I(\theta) = \underbrace{\left| \overset{\mathrm{v}}{\nabla}|_\theta u \wedge \underline{v} \right|^2}_{= \left| \overset{\mathrm{v}}{\nabla}|_\theta u \right|^2} \cdot \underbrace{K\left(\overset{\mathrm{v}}{\nabla}|_\theta u, \underline{v} \right)}_{\leq -\kappa \text{ by hypothesis}}$, which gives:

$$\begin{aligned} &\leq -\kappa \iint \left| \overset{\mathrm{v}}{\nabla}|_\theta u \right|^2 \\ &= -\kappa \left\| \overset{\mathrm{v}}{\nabla} u \right\|_{\mathcal{Z}}^2 \end{aligned}$$

Step 3:

$$
\begin{aligned}
-\kappa \left\| \overset{\text{v}}{\nabla} u \right\|_{\mathcal{Z}}^2 &= -\kappa \left(\overset{\text{v}}{\nabla} u, \overset{\text{v}}{\nabla} u \right)_{\mathcal{Z}} \\
&= -\kappa \left(\overset{\text{v}}{\nabla}^* \overset{\text{v}}{\nabla} u, u \right)_{L^2(SM)} \\
&= -\kappa \left(-\Delta u, u \right) \\
&= -\kappa \left(\lambda_l u, u \right) \\
&= -\kappa \lambda_l \left(u, u \right) \\
&= -\kappa \lambda_l \left\| u \right\|^2
\end{aligned}
$$

4.6 Proof of Proposition 6.6

Proposition 6.6 is a precursor to the Carleman Estimate. Essentially, we set up an estimate that works for *finite-degree* functions, and then we vary the coefficients/weights in the estimate to force it to hold for *arbitrary* functions as well.

Start with the single-frequency estimates from 6.4. The upper bound is hard to work with; would prefer an upper bound in terms of $\|(Xu)_{l+1}\|^2$ instead. Write the current upper bound as the sum of our desired upper bound and an error term. We notice that part of the error term can be absorbed into the LHS when we move it over. We'll use this to our advantage: rather than estimating away the entire error term, which would just introduce more garbage for us to deal with, we estimate away only *part* of the error term: more specifically, we use the "ok" part of the error term to estimate away the rest - using the $x + 1/x$ trick. That way, we're not introducing anything new. In place of the error term is something that can still be entirely absorbed into the LHS.

Outline

1. Take the estimates from 6.4 and multiply them by the corresponding terms of a sequence $\{\gamma_l^2\}$. Turn the RHS into desired + error We'll impose conditions on the sequence in the course of the proof
2. estimate away the bad part of the error
3. sum up all the localised estimates and absorb
4. rigging - second bound on epsilon and condition on gamma
5. Nail down the allowable values of the ε's
6. Take the limit

Proof

Step 1:
To begin, apply Lemma 6.4 to each component of u and multiply through by γ_l^2:

$$\alpha_{l-1}\gamma_l^2 \|X_- u_l\|^2 + \kappa\lambda_l\gamma_l^2 \|u_l\|^2 + \gamma_l^2 \|Z(u_l)\|^2 \leq \beta_{l+1}\gamma_l^2 \|X_+ u_l\|^2$$

Using the fact that $(Xu)_{l+1} = X_+ u_l + X_- u_{l+2}$, we can express the RHS in terms of our desired quantity. Rearrange, take norms, and expand to get our desired RHS + error:

$$\begin{aligned}\|X_+ u_l\|^2 &= \|(Xu)_{l+1} - X_- u_{l+2}\|^2 \\ &= \underbrace{\|(Xu)_{l+1}\|^2}_{\text{desired RHS}} \underbrace{-2\mathrm{Re}\left((Xu)_{l+1}, X_- u_{l+2}\right)}_{\text{“bad” part of the error}} \underbrace{+ \|X_- u_{l+2}\|^2}_{\text{“ok” part of the error}}\end{aligned}$$

Step 2:

We'll now estimate the above expression in terms of only the two outer terms. Consider the following, as a function of ε_l:

$$(1+\varepsilon_l^{-1})\left\|(Xu)_{l+1}\right\|^2+(1+\varepsilon_l)\left\|X_-u_{l+2}\right\|^2$$

It has the form

$$\begin{aligned}&(1+\varepsilon^{-1})A+(1+\varepsilon)C\\=&A+C+A/\varepsilon+C\varepsilon\end{aligned}$$

On the other hand, the thing we're trying to estimate has the form $A-2B+C$. In order for our estimate to work, we need to be able to choose ε so that $A/\varepsilon+C\varepsilon\geq-2B$, regardless of how large $-2B$ is (it can be positive). Now, $A/\varepsilon+C\varepsilon$ behaves just like $x+1/x$, which gets arbitrarily large in both directions on $(0,\infty)$, so we're fine. To be specific, the allowable values of ε are:

Case 1: $-2B\leq\min_{\varepsilon>0}A/\varepsilon+C\varepsilon$
In this case, any ε will work.

Case 2: $-2B>\min_{\varepsilon>0}A/\varepsilon+C\varepsilon$
The minimum is $2\sqrt{AC}$, which occurs at the value $\varepsilon=\sqrt{A/C}$. Solving for ε in the case of equality gives $\frac{B\pm\sqrt{B^2-AC}}{C}$; letting these two numbers be q and Q, respectively, the values that ε can take in this case are $(0,q]\cup[Q,\infty)$.

Here's where things get a little hairy: this is all fine and good as long as A, B, and C are constants. But if you go back to where I introduced them, they actually depend on u (they also depend on l, but that doesn't introduce any complications, so we'll continue to suppress l for now). Hence the allowable values of ε actually depend on u. We want, however, *one* value of ε that works for all u. These are:

$$\begin{aligned}&\bigcap_{u\in\text{Case 2}}(0,q(u)]\cup[Q(u),\infty)\\=&\left(0,\inf_{u\in\text{Case 2}}q(u)\right]\cup\left[\sup_{u\in\text{Case 2}}Q(u),\infty\right)\end{aligned}$$

The obvious first question is, "Is the above set even nonempty?" We'll address this question in Step 5. For now, assume that some ε exists.

We now have:

$$\alpha_{l-1}\gamma_l^2\left\|X_-u_l\right\|^2+\kappa\lambda_l\gamma_l^2\left\|u_l\right\|^2+\gamma_l^2\left\|Z(u_l)\right\|^2\leq\beta_{l+1}\gamma_l^2(1+\varepsilon_l^{-1})\left\|(Xu)_{l+1}\right\|^2+\beta_{l+1}\gamma_l^2(1+\varepsilon_l)\left\|X_-u_{l+2}\right\|^2$$

Step 3:

Summing up the estimates for all degrees m to N and reindexing the sums on the RHS so that the indices inside norms are all l's, we get something like:

$$\sum_{l=m}^{N} a_l \|X_- u_l\|^2 + \sum_{l=m}^{N} b_l \|u_l\|^2 + \sum_{l=m}^{N} c_l \|Z(u_l)\|^2 \leq \sum_{l=m+1}^{N+1} d_l \|(Xu)_l\|^2 + \sum_{l=m+2}^{N+2} e_l \|X_- u_l\|^2$$

Now comes the point where we absorb: Notice that there's an X_- term on either side. We want to subtract the one on the RHS from the one on the LHS and rig the a's and e's so that the result is still positive for the indices that "overlap". Notice, however, that the $(N+1)$- and $(N+2)$-terms on the RHS don't overlap with anything on the LHS, and they become negative when we move them over. This isn't as problematic as it may seem at first: recall that we're only focusing on u with finite degree right now. So as long as $N \geq \deg(u)$, we're fine: those two pesky terms vanish on their own. We end up with:

$$\sum_{l=m}^{N+2} \left\{ \begin{array}{cc} a_l & l = m, m+1 \\ a_l - e_l & l = m+2, \ldots, N \\ -e_l & l = N+1, N+2 \end{array} \right\} \cdot \|X_- u_l\|^2 + \sum_{l=m}^{N} b_l \|u_l\|^2 + \sum_{l=m}^{N} c_l \|Z(u_l)\|^2 \leq \sum_{l=m+1}^{N+1} d_l \|(Xu)_l\|^2$$

<u>Step 4:</u>
As discussed, for this estimate to end up doing what we want, we need $a_l - e_l \geq 0$. In terms of the actual coefficients, this places the following restriction on the sequence of ε's:

$$\varepsilon_l \leq \underbrace{\left(\frac{\alpha_{l+1} \gamma_{l+2}^2}{\beta_{l+1} \gamma_l^2} \right) - 1}_{=: M_l} \qquad \text{for } l = m, \cdots, N-2$$

Because we're going to end up taking $N- > \infty$, we'll further impose that the above holds for all l greater than m. A note on notation: notice that M_l depends on the sequence $\{\gamma_l\}$; when we wish to emphasize this dependence, we'll write $M_l(\gamma)$ rather than $M_l(\{\gamma_l\})$ to avoid anyy confusion caused by the l on the γ.

Vis-a-vis this bound, there are two things to consider:

1. We obviously need $M_l > 0$, and this places a condition on the sequence $\{\gamma_l\}$, which to this point had no restrictions on it.
2. We need to ensure that it jives with the condition on ε from Step 2. This will be addressed in step 5.

(1) It also places a restriction on the sequence $\{\gamma_l\}$ places the following restriction on the sequence of γ's (it's essentially a growth condition):

$$\alpha_l \gamma_{l+1}^2 > \beta_l \gamma_{l-1}^2 \qquad \text{for } \; l \geq m+1$$

<u>Step 5:</u>

Now we come to the task of determining the allowable values of the ε's. The following intersection combines the restriction from Step 2 with the restriction from Step 5:

$$\varepsilon_l \in (0, M_l] \cap \left\{ \left(0, \inf_{u \in \text{Case } 2(l)} q_l(u) \right] \cup \left[\sup_{u \in \text{Case } 2(l)} Q_l(u), \infty \right) \right\}$$

When we use the inequality from the statement of the theorem to prove Theorem 6.1, we'll want to take $\varepsilon_l = M_l \ \forall l$, so we need to show that:

$$M_l \leq \inf_{u \in \text{Case } 2(l)} q_l(u) \qquad \text{for } l = m, \cdots, N-2$$

While we will be fixing $\{\gamma_l\}$ when we apply the inequality in the proof of Theorem 6.1, the statement of the present theorem allows $\{\gamma_l\}$ to be arbitrary, so long as it satisfies the growth condition. Hence we need to prove the above without knowing anything about $\{\gamma_l\}$ other than that it satisfies the growth condition. An equivalent way to state this is that we must prove:

$$\sup_{\substack{\gamma \text{ satisfying the} \\ \text{growth condition}}} M_l(\gamma) \leq \inf_{u \in \text{Case } 2(l)} q_l(u) \qquad \text{for } l = m, \cdots, N-2$$

None of this is addressed in the paper, which to me is a major plothole, because I have no idea how I would even begin to prove such a thing - or any idea why I should expect it to be true.

To make things more compact, we'll reparametrize the intervals $(0, M_l]$ by $(0, 1]$ as follows: by replacing ε_l by $\delta_{l+1} M_l$, where the δ's can take any value in $(0, 1]$ (the index is different on the δ's only so that ...), we don't have to remember the allowable range of each ε_l – or even that they depend on $\{\gamma_l\}$ – because it's encoded in the inequality; we just have to remember the easy requirement that $\delta_l \in (0, 1] \ \forall l$.

we need to make sure that they don't contradict each other. Or, more accurately, we need to make sure that it's possible to choose the γ's – which are as yet variable – in a way to make the bound from the present step jive with the condition from Step 2 – which is fixed.

Appendix A

Reference Materials

A.1 Notation

A note on how this list is organized: rather than doing it alphabetically, I decided to organize it hierarchically. This, in my opinion, gives the added benefit of using this not only as a list of notation, but also as a way of collecting up all the objects considered in the paper in a way that the reader can use it to see how they all relate to each other.

Miscellaneous

$\langle l \rangle$ - $(1 + l^2)^{1/2}$. This shorthand is called the **Japanese angle bracket**

Geometric objects

M - Smooth manifold with boundary

- d - the dimension of M
- x - the general point of M
- $\underline{v}$ - the general tangent vector of M; based at x
- φ - a coordinate patch on M

SM - sphere bundle of (M, g)

- θ - general point of SM. That is, $\theta = (x; \underline{v})$
- ξ - the general tangent vector of SM; based at θ
- π - the bundle projection $SM \to M$
- $\partial_+(SM)$ - inward-pointing unit tangent vectors on ∂M
- $\partial_-(SM)$ - outward-pointing unit tangent vectors on ∂M
- $\partial_0(SM)$ - the unit vectors that are tangent to ∂M

γ_θ - the geodesic in M determined by θ (that is, starting at x and going in the direction $\underline{v}$)

- $\tau(\theta)$ - the time at which the geodesic γ_θ arrives at ∂M

ϕ_t^X - geodesic flow of M. We look at it as living on SM rather than the full TM

$\mathcal{K}_\theta$ - the connection map coming from the Levi-Civita connection associated to the Sasaki metric

$\mathcal{A}$ - an arbitrary attenuation for the ray transform (not necessarily linear)

- a - a scalar attenuation (used in the ray transform on functions)
- A - general connection (linear)
- Φ - general Higgs field

Vector fields and other tensors

T - general (covariant) tensor field to which we apply the ray transform

$\widetilde{T}$ - the function on SM extracted from T

X - geodesic vector field of (M, g). We conceptualize it as living on SM (i.e. it's a section of $T(SM)$)

$\mathcal{V}$ - vertical vector field on SM

$\mathcal{H}$ - the horizontal vector field on SM that's induced by the Sasaki metric

R - Riemann curvature of M. Written $R_x[\underline{v}_1, \underline{v}_2](\underline{w})$

- $\widetilde{R}$ - a field of operators on the bundle N coming from R. Defined $\widetilde{R}_\theta := R_x[-, \underline{v}](\underline{v})$. [Lee09] refers to this as the **tidal operator** (p.558)

∂_j - essentially just ∂_{y_j}, but for functions defined on SM only

δ_{x_j} - $\partial_{x_j} - (\sum_{k,l})\Gamma^l_{jk} y^k \partial_{y_l}$

δ_j - essentially just δ_{x_j}, but for functions defined on SM only

Function spaces

$L^2(SM, \mathbb{C}^n)$ -

u - the general function in $L^2(SM, \mathbb{C}^n)$

u_l - components of u w.r.t. the vertical spherical harmonics decomposition

m - the degree of finite-degree functions in $L^2(SM, \mathbb{C}^n)$

$C^\infty(SM, \mathbb{C}^n)$ -

$C^\infty_F(SM, \mathbb{C}^n)$ - (from [PS18]; I don't use this notation) smooth functions of finite degree (i.e. polynomials on SM)

$\mathscr{P}_m, \mathscr{P}'_m$ - homogeneous polynomials of degree m on TM, SM respectively

$H_m(SM)$ - vertical spherical harmonics of degree m

λ_m - the eigenvalue corresponding to $H_m(SM)$. It $= m(m+d-2)$

Ω_m - **smooth** vertical spherical harmonics of degree m, for $m \geq 0$. For convenience, we define $\Omega_{-1} \coloneqq \{0\}$

$\mathscr{Z}$ - TM-valued functions on SM with certain properties

Z - the general function in $\mathscr{Z}^n$

Inner Products

g - Riemannian metric on M

$\langle -, - \rangle$ - alternative notation for g

$\langle\langle -, - \rangle\rangle$ - the Sasaki metric on SM

$(-, -)$ - will be used for the inner products of both $L^2(SM, \mathbb{C}^n)$ and $\mathcal{Z}^n$. In the text, we will distinguish as follows: absence of a subscript indicates $L^2(SM, \mathbb{C}^n)$, while the $\mathcal{Z}^n$ inner product will always have a subscript $\mathcal{Z}$

$(-, -)_\alpha$ - modified inner product on $L^2(SM, \mathbb{C}^n)$, where α is a sequence of complex numbers. Note that a Greek letter in the subscript automatically implies that this is an inner product on $L^2(SM, \mathbb{C}^n)$. Such an object is not defined on $\mathcal{Z}^n$

Operators

—	—	Domain(s)	Codomain(s)	Action	[PS18]
d	inner derivative	symmetric k-tensors	symmetric $(k-1)$-tensors		
$\widetilde{A}$	general connection	$C^\infty(SM, \mathbb{C}^n)$	$C^\infty(SM, \mathbb{C}^n)$	matrix multiplication	p.17
$\widetilde{R}$		$\mathcal{Z}^n$	$\mathcal{Z}^n$		p.11
X	geodesic vector field	$C^\infty(SM, \mathbb{C}^n)$	$C^\infty(SM, \mathbb{C}^n)$	$X\|_\theta u = \frac{d}{dt}\|_0 u(\phi_t^X(\theta))$	p.2
—	—	$\mathcal{Z}^n$	$\mathcal{Z}^n$	$X\|_\theta Z = \frac{D}{dt}\|_0 Z(\phi_t^X(\theta))$	p.11
X_+		Ω_m	Ω_{m+1}		p.11
X_-		Ω_m	Ω_{m-1}		p.11
X^A	$X + \widetilde{A}$	$C^\infty(SM, \mathbb{C}^n)$	$C^\infty(SM, \mathbb{C}^n)$		p.17
Δ	vertical Laplacian	$C^\infty(SM, \mathbb{C}^n)$	$C^\infty(SM, \mathbb{C}^n)$	$\overset{\mathrm{v}}{\mathrm{div}}\overset{\mathrm{v}}{\nabla}$	
∇^{SM}	total derivative				
$\overset{\mathrm{v}}{\nabla}$	vertical gradient	$C^\infty(SM, \mathbb{C}^n)$	$\mathcal{Z}^n$		p.11
$-\overset{\mathrm{v}}{\mathrm{div}}$	adjoint of $\overset{\mathrm{v}}{\nabla}$ w.r.t $(-,-)_{\mathcal{Z}}$	$\mathcal{Z}^n$	$C^\infty(SM, \mathbb{C}^n)$		
$\overset{\mathrm{h}}{\nabla}$	horizontal gradient	$C^\infty(SM, \mathbb{C}^n)$	$\mathcal{Z}^n$		p.11
$-\overset{\mathrm{h}}{\mathrm{div}}$	adjoint of $\overset{\mathrm{h}}{\nabla}$ w.r.t $(-,-)_{\mathcal{Z}}$	$\mathcal{Z}^n$	$C^\infty(SM, \mathbb{C}^n)$		
$\overset{\mathrm{h}}{\nabla}_A$	$\overset{\mathrm{h}}{\nabla} + \overset{\mathrm{v}}{\nabla}\widetilde{A}$	$C^\infty(SM, \mathbb{C}^n)$	$\mathcal{Z}^n$		p.17
$\overset{\mathrm{h}}{\mathrm{div}}_A$	$\overset{\mathrm{h}}{\mathrm{div}} + \left\langle \overset{\mathrm{v}}{\nabla}\widetilde{A}, \cdot \right\rangle$	$\mathcal{Z}^n$	$C^\infty(SM, \mathbb{C}^n)$		p.17
F^A	$X\overset{\mathrm{v}}{\nabla}\widetilde{A} - \overset{\mathrm{h}}{\nabla}\widetilde{A} + [\widetilde{A}, \overset{\mathrm{v}}{\nabla}\widetilde{A}]$	$C^\infty(SM, \mathbb{C}^n)$	$\mathcal{Z}^n$		p.17
Z, Z_A	$\overset{\mathrm{v}}{\mathrm{div}}$-free part of $\overset{\mathrm{h}}{\nabla}_A u$	$C^\infty(SM, \mathbb{C}^n)$	$\mathcal{Z}^n$		p.18
P	$\overset{\mathrm{v}}{\nabla}X$	$C^\infty(SM, \mathbb{C}^n)$	$\mathcal{Z}^n$		p.13

Sequences of Weights

$\{\alpha_l\}$ - $\begin{cases} d-1 & l=0 \\ (2l+d-2)\left(1+\frac{1}{l+d-2}\right) & l \geq 1 \end{cases}$ (p.18)

$= 2(l+1) + (d-2)(1-{}^{1}\!/_{l}) + \frac{(d-2)^2}{l(l+d-2)}$ (p. 30)

$= 2l + d - \frac{d-2}{l+d-2}$ (mine)

$\{\beta_l\}$ - $\begin{cases} 0 & l=0,1 \\ (2l+d-2)\,(1-1/l) & l \geq 2 \end{cases}$ (p.18)

$= 2(l-1) + (d-2)(1-{}^{1}\!/_{l})$ for $l \geq 2$ (p.30)

$= 2l + d - 4 - \frac{d-2}{l}$ (mine)

$\{\gamma_l\}$ - appears in Proposition 6.6; see 4.3 for choice in proving the Carleman estimate

$\{\delta_l\}$ - appears in Proposition 6.6; see 4.3 for choice in proving the Carleman estimate

$\{\varepsilon_l\}$ - auxiliary sequence appearing in the proof of Proposition 6.6

$\{r_l\}$ - auxiliary sequence appearing in the derivation of the choice of weights (4.3); $r_l := l\gamma_l^2$

The following table records the first few values of $\{\alpha_l\}$ for various values of d and m:

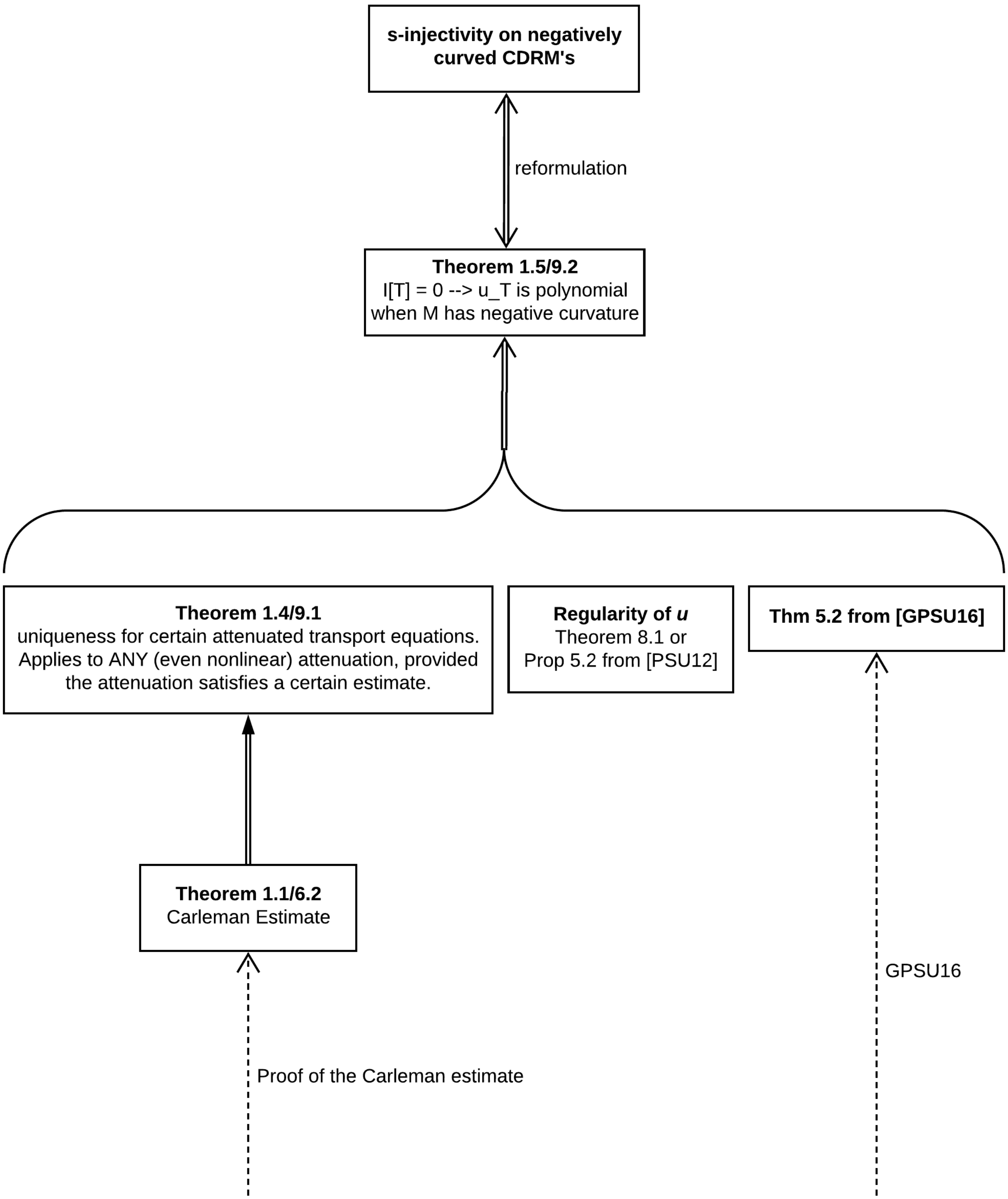
$d \setminus m$	0	1	2	3	4	5	6	7	8	9	10
2	1	4	6	8	10	12	14	16	18	20	22
3	2	$4\frac{1}{2}$	$6\frac{2}{3}$	$8\frac{3}{4}$	$10\frac{4}{5}$	$12\frac{5}{6}$	$14\frac{6}{7}$	$16\frac{7}{8}$	$18\frac{8}{9}$	$20\frac{9}{10}$	$22\frac{10}{11}$
4	3	$5\frac{1}{3}$	$7\frac{2}{4}$	$9\frac{3}{5}$	$11\frac{4}{6}$	$13\frac{5}{7}$	$15\frac{6}{8}$	$17\frac{7}{9}$	$19\frac{8}{10}$	$21\frac{9}{11}$	$23\frac{10}{12}$
5	4	$6\frac{1}{4}$	$8\frac{2}{5}$	$10\frac{3}{6}$	$12\frac{4}{7}$	$14\frac{5}{8}$	$16\frac{6}{9}$	$18\frac{7}{10}$	$20\frac{8}{11}$	$22\frac{9}{12}$	$24\frac{10}{13}$
6	5	$7\frac{1}{5}$	$9\frac{2}{6}$	$11\frac{3}{7}$	$13\frac{4}{8}$	$15\frac{5}{9}$	$17\frac{6}{10}$	$19\frac{7}{11}$	$21\frac{8}{12}$	$23\frac{9}{13}$	$25\frac{10}{14}$

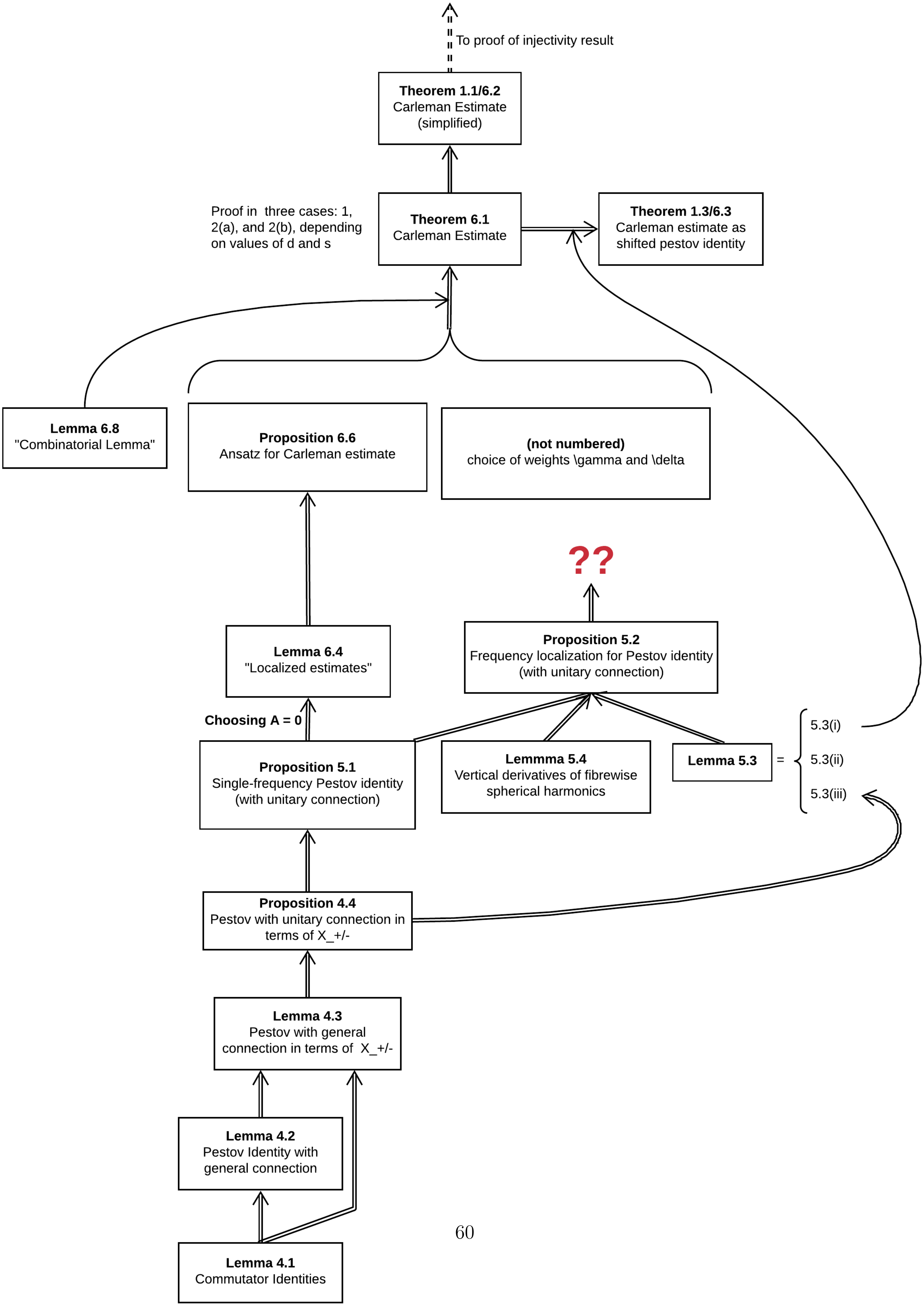
The following table records the first few values of $\{\beta_l\}$ for various values of d and m:

$d \setminus m$	0	1	2	3	4	5	6	7	8	9	10
2	0	0	2	4	6	8	10	12	14	16	18
3	0	0	$2\frac{1}{2}$	$4\frac{2}{3}$	$6\frac{3}{4}$	$8\frac{4}{5}$	$10\frac{5}{6}$	$12\frac{6}{7}$	$14\frac{7}{8}$	$16\frac{8}{9}$	$18\frac{9}{10}$
4	0	0	3	$5\frac{1}{3}$	$7\frac{2}{4}$	$9\frac{3}{5}$	$11\frac{4}{6}$	$13\frac{5}{7}$	$15\frac{6}{8}$	$17\frac{7}{9}$	$19\frac{8}{10}$
5	0	0	$3\frac{1}{2}$	6	$8\frac{1}{4}$	$10\frac{2}{5}$	$12\frac{3}{6}$	$14\frac{4}{7}$	$16\frac{5}{8}$	$18\frac{6}{9}$	$20\frac{7}{10}$
6	0	0	4	$6\frac{2}{3}$	9	$11\frac{1}{5}$	$13\frac{2}{6}$	$15\frac{3}{7}$	$17\frac{4}{8}$	$19\frac{5}{9}$	$21\frac{6}{10}$

- the prototype for the α's is $x - \frac{d}{x+d}$, and the prototype for the β's is $x - \frac{d}{x}$
- both the α's and β's are asymptotically linear (with slope always 2) for all values of d
- as d increases, the α's become more linear, whereas the β's become less linear (i.e they take longer to converge)
- the other thing that occurs as d increases is that all the values of both sequences increase at the same rate

A.2 Overview/Roadmap (Flowcharts)




To proof of injectivity result
Theorem 1.1/6.2
Carleman Estimate
(simplified)
Proof in three cases: 1, 2(a), and 2(b), depending on values of d and s
Theorem 6.1
Carleman Estimate
Theorem 1.3/6.3
Carleman estimate as shifted pestov identity
Lemma 6.8
"Combinatorial Lemma"
Proposition 6.6
Ansatz for Carleman estimate
(not numbered)
choice of weights \gamma and \delta
??
Lemma 6.4
"Localized estimates"
Proposition 5.2
Frequency localization for Pestov identity
(with unitary connection)
Choosing A = 0
Proposition 5.1
Single-frequency Pestov identity
(with unitary connection)
Lemmma 5.4
Vertical derivatives of fibrewise spherical harmonics
Lemma 5.3
=
5.3(i)
5.3(ii)
5.3(iii)
Proposition 4.4
Pestov with unitary connection in terms of X_+/-
Lemma 4.3
Pestov with general connection in terms of X_+/-
Lemma 4.2
Pestov Identity with general connection
Lemma 4.1
Commutator Identities

A.3 Theorems

		page	page in [PS18]
	Reformulation	12	-
Theorem 1.1/6.2	Carleman estimate	12	3/24
Theorem 1.4/9.1		12	5/40
Theorem 1.5/9.2	Main result	13	6/41
Theorem $GPSU16-5.2$	Vanishing criterion	62	-
Theorem 6.1		35	23
Lemma 6.4	Localized estimates	35	25
Proposition 6.6		35	26
Lemma 6.8	Combinatorial lemma	63	28
Proposition 5.1	Single-frequency Pestov identity	23	20
Lemma 5.2	Frequency-localized Pestov identity	24	21
Lemma 5.3		25	21
Lemma 4.1	Commutator identities	32	17
Lemma 4.2	Pestov with general connection		17
Lemma 4.3	Pestov with general connection in terms of $X_\pm$		18
Proposition 4.4	Pestov with unitary connection in terms of $X_\pm$		20

Theorem (Reformulation). The ray transform on M is s-injective $\iff$ for all polynomials f, we have $\partial_+ u_f \equiv 0 \implies u_f$ is polynomial.

Theorem 1.1/6.2 (The Carleman Estimate). On a compact Riemannian manifold with negative sectional curvature $\leq -\kappa < 0$ for some $\kappa > 0$ (compact $\implies$ bounded away from 0), the following holds for any $\tau \geq 1$ and $m \in \mathbb{Z}_+$:

$$\sum_{l=m}^{\infty} e^{2\tau\varphi_l} \|u_l\|^2 \leq \frac{(d+4)^2}{\kappa\tau} \sum_{l=m+1}^{\infty} e^{2\tau\varphi_l} \|(Xu)_l\|^2$$

where:

- $u \in C^\infty(SM)$, with $u|_{\partial SM} \equiv 0$ if M has boundary
- X is the geodesic vector field (acting as a differential operator)
- l as subscript refers to decomp w.r.t. vertical spherical harmonics
- $\|\cdot\|$ is the norm on $L^2(SM)$
- $\varphi_l = \log(l)$
- $d = \dim(M)$

Theorem 1.4/9.1.

(i) Under the same hypotheses as Theorem 1.1/6.2, it's known that smooth solutions to the attenuated transport equation $Xu + \mathcal{A}(u) = -f$ (with boundary condition $\partial u \equiv 0$) have finite degree when:

- f itself is smooth and has finite degree (i.e. $f \in C^\infty_F(SM, \mathbb{C}^n)$)

- $\mathcal{A}$ is any operator on $C^\infty(SM, \mathbb{C}^n)$ that satisfies $\|(\mathcal{A}(u))_l\| \leq R \cdot (\|u_{l-l}\| + \|u_l\| + \|u_{l+1}\|)$ for $l \geq$ some $l_0 \geq 2$

(ii) $\deg(f) \leq \max\{l_0 - 1, \deg(f), 2C_{d,\kappa}R\} - 1$
- $C_{d,\kappa}$ is defined through the course of the proof

Theorem 1.5/9.2 (Main theorem of PS18).

(i) Under the following conditions, $u_f^{A+\Phi}$ is a polynomial of degree $\leq \deg(f) - 1$:
- M satisfies the hypotheses of Theorem 1.1/6.2 and is non-trapping
- <u>either</u> ∂M is strictly convex <u>or</u> $\operatorname{supp}(f) \subset SM^\circ$
- f is a polynomial
- $f \in \ker I^{A+\Phi}$

(ii) In the case that f is homogeneous (such as when $f = \widetilde{T}$), u_f is also homogeneous, and its degree is exactly $\deg(f) - 1$

Theorem GPSU16-5.2 ("Vanishing Criterion"). If $u \in \Omega_m^{\mathcal{E}}$ - where $\mathcal{E}$ is a Hermitian bundle over SM with a (Hermitian) connection - satisfies the following two conditions, then, in fact, u vanishes identically on all of SM.
- u vanishes on the preimage (under π_{SM}) of some hypersurface $\Gamma \subset M$
- u is killed by $\mathbb{X}_+ := (\nabla_X^{\mathcal{E}})_+$

Theorem 6.1. Under the same conditions as Theorem 1.1/6.2, the following holds for $s > -1/2$ and $m \in \mathbb{Z}_+$:

$$\sum_{l=m}^{\infty} \left\{ \begin{array}{cc} 2l^{2s+1} & l = m, m+1 \\ (2s+1)(l-1)^{2s} & l \geq m+2 \end{array} \right\} \|X_- u_l\|^2 + \kappa \sum_{l=m}^{\infty} l^{2s+2} \|u_l\|^2 + \sum_{l=m}^{\infty} l^{2s} \|Z(u_l)\|_{\mathcal{Z}}^2$$
$$\leq C \sum_{l=m+1}^{\infty} l^{2s+2} \|(Xu)_l\|^2$$

- See the discussion after the proof for values of the constant C

Lemma 6.4 ("Localised Estimates"). Under the conditions of Theorem 1.1/6.2, the following holds for $l \geq 0$:

$$\alpha_{l-1} \|X_- u\|^2 + \kappa\lambda_l \|u\|^2 + \|Z(u)\|^2 \leq \beta_{l+1} \|X_+ u\|^2$$

- $u \in \Omega_l$

Proposition 6.6 (Ansatz for Carleman estimate). Under the conditions of Theorem 1.1/6.2, the following holds for functions of *finite degree*:

$$\sum_{l=m}^{\infty} \left\{ \begin{array}{cc} \alpha_{l-1}\gamma_l^2 & l = m, m+1 \\ (1-\delta_{l-1})(\alpha_{l-1}\gamma_l^2 - \beta_{l-1}\gamma_{l-2}^2) & l \geq m+2 \end{array} \right\} \|X_- u_l\|^2 + \kappa \sum_{l=m}^{\infty} \lambda_l \gamma_l^2 \|u_l\|^2 + \sum_{l=m}^{\infty} \gamma_l^2 \|Z(U)\|_{\mathcal{Z}}^2$$
$$\leq \sum_{l=m+1}^{\infty} \left(1 + \frac{1-\delta_l}{\delta_l} \cdot \frac{\beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2}\right) \frac{\alpha_l \gamma_{l+1}^2 \beta_l \gamma_{l-1}^2}{\alpha_l \gamma_{l+1}^2 - \beta_l \gamma_{l-1}^2} \|(Xu)_l\|^2$$

- m is any integer ≥ 1
- $\{\gamma_l\}$ is any sequence in $\mathbb{R}_+$ that satisfies $\alpha_l\gamma_{l+1}^2 > \beta_l\gamma_{l-1}^2$ for $l \geq m+1$
- $\{\delta_l\}$ is *any* sequence in $(0,1]$
- u is a polynomial (smooth and finite degree)
- $u|_{\partial SM} \equiv 0$ if M has boundary

Remark: the infinitude sums is somewhat artificial, since they always reduce to finite sums when you plug in u.

Lemma 6.8 ("Combinatorial Lemma").

(i) $(l+1)^p - (l-1)^p \geq pl^{p-1}$ for $p \geq 0, l \geq 1$

(ii) $(l+1)^p - (l-1)^p \geq -\eta_p(l_0)l^{p-1}$ for $p \in (-1,0), l \geq l_0 \geq 2$

- They use the letter s, which conflicts with the s in the statement and proof of Thm 6.1. I've switched to p to avoid the conflict
- Proof omitted (no insight to be gained)

Proposition 5.1 (Single-Frequency Pestov Identity).

$$\alpha_{l-1}\left\|X_-^A u\right\|^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u, \overset{\mathrm{v}}{\nabla}u\right)_{\mathcal{Z}} - \left(F_A(u), \overset{\mathrm{v}}{\nabla}u\right)_{\mathcal{Z}} + \|Z_A(u)\|_{\mathcal{Z}}^2 = \beta_{l+1}\left\|X_+^A u\right\|^2$$

- M compact; with or without boundary
- A a unitary connection
- $l \geq 0$
- $u \in \Omega_l$
- $u|_{\partial SM} \equiv 0$ if M has boundary
- α_l, β_l as defined in XXX

Lemma 5.2 (Frequency-Localised Pestov Identity). For $u \in C^\infty(SM)$ and unitary connection A we have:

$$\begin{aligned}
&\left\|X_-^A u\right\|_{\alpha\ (L^2(SM))}^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u, \overset{\mathrm{v}}{\nabla}u\right)_{\mathcal{Z}} - \left(F_A u, \overset{\mathrm{v}}{\nabla}u\right)_{\mathcal{Z}} + \|Z_A(u)\|_{\mathcal{Z}}^2 \\
=&\sum_{l=0}^{\infty}\left[\alpha_{l-1}\left\|X_-^A u_l\right\|^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla}u_l, \overset{\mathrm{v}}{\nabla}u_l\right)_{\mathcal{Z}} - \left(F_A(u_l), \overset{\mathrm{v}}{\nabla}u_l\right)_{\mathcal{Z}} + \|Z_A(u_l)\|_{\mathcal{Z}}^2\right] \\
=&\sum_{l=0}^{\infty}\beta_{l+1}\left\|X_+^A u_l\right\|^2 \\
=&\left\|X_+^A u\right\|_{\beta\ (L^2(SM))}^2
\end{aligned}$$

- **Note:** the statement of Lemma 5.2 doesn't actually say what they want it to say; as it's written there, they subract the RHS's from the LHS's *within* each summand, so that all summands reduce to 0 (as does the RHS). Hence their equation reads $\sum 0 = 0$, which doesn't say anything meaningful. The proper way to write it out is to have a four-way equality as presented here

Lemma 5.3. For $u \in C^\infty(SM)$ and unitary connection A we have:

(i) $\left(\widetilde{R}\overset{\mathrm{v}}{\nabla} u_l, \overset{\mathrm{v}}{\nabla} u_m\right)_{\mathcal{Z}} = 0$

(ii) $\left(F^A u_l, \overset{\mathrm{v}}{\nabla} u_m\right)_{\mathcal{Z}} = 0$

(iii) $(Z_A(u_l), Z_A(u_m))_{\mathcal{Z}} = 0$

Lemma 4.1 (Commutator identities).

(i) $[X^A, \overset{\mathrm{v}}{\nabla}] = -\overset{\mathrm{h}}{\nabla}_A$

(ii) $[X^A, \overset{\mathrm{h}}{\nabla}_A] = \widetilde{R}\overset{\mathrm{v}}{\nabla} + F^A$

(iii) $\overset{\mathrm{h}}{\mathrm{div}}_A \overset{\mathrm{v}}{\nabla} - \overset{\mathrm{v}}{\mathrm{div}} \overset{\mathrm{h}}{\nabla}_A = (d-1)X^A$

(iv) $[X^A, \overset{\mathrm{v}}{\mathrm{div}}] = -\overset{\mathrm{h}}{\mathrm{div}}_A$

The first three are on $C^\infty(SM, \mathbb{C}^n)$, the last is on $\mathcal{Z}^n$

Lemma 4.2 (Pestov for general connection A).

$$\left(\overset{\mathrm{v}}{\nabla} X^A u, \overset{\mathrm{v}}{\nabla} X^{-A^*} u\right)_{\mathcal{Z}} = \left(X^A \overset{\mathrm{v}}{\nabla} u, X^{-A^*} \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F^A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + (d-1)\left(X^A u, X^{-A^*}\right)$$

- They note that the non-unitarity of A means a loss of symmetry in the Pestov identity, which is seen here

Lemma 4.3 (Pestov in terms of $X_\pm^A$, n.n. unitary).

$$\left(X_-^A u, X_-^{-A^*} u\right)_\alpha - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F^A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + (Z_A(u), Z_{-A^*}(u))_{\mathcal{Z}} = \left(X_+^A u, X_+^{-A^*} u\right)_\beta$$

Proposition 4.4 (Pestov Identity in terms of $X_\pm^A$, A unitary).

$$\left\|X_-^A u\right\|_\alpha^2 - \left(\widetilde{R}\overset{\mathrm{v}}{\nabla} u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} - \left(F^A u, \overset{\mathrm{v}}{\nabla} u\right)_{\mathcal{Z}} + \|Z_A(u)\|_{\mathcal{Z}}^2 = \left\|X_+^A u\right\|_\beta^2$$

Appendix B

Technical Preliminaries

B.1 "Polynomials" on TM

Definition. A **polynomial function on** TM is a function on TM whose expression in coordinates is polynomial in the coordinates of $\underline{v}$ with coefficients in $C^\infty(M)$.

1. The definition doesn't depend on choice of coordinates, since coordinates change linearly on tangent spaces; replacing each variable with a linear combination of the others is still a polynomial
2. Notice that we don't ask for the expression to be polynomial in the coordinates of x, however; that property could very well hold in one chart and not in another. So basically "polynomial" on TM means "polynomial in all the variables for which it makes sense to talk about such a thing".

Notice that the lift (as defined in Section 1.1) of any (covariant) tensor is a polynomial. For example, say M is 3-dimensional and we have a 3-tensor $T = T_{113}(x) \cdot dx^1 \otimes dx^1 \otimes dx^3$; the corresponding polynomial function is $\widetilde{T}(\theta) = T_{113}(x) \cdot v_1^2 v_3$. Moreover, any polynomial can be obtained from lifting a tensor: just declare its coefficients to be the components of a tensor (in the same basis).

- in a corresponding tensor/polynomial pair, the polynomial is homogeneous iff the tensor is homogeneous.
- we're going to restrict everything to be homogeneous because inhomogeneous tensors don't really come up in pure geometry (more like algebraic topology)
- the lifting operation is not injective in general
- but it *is* injective on symmetric tensors: given a homogeneous polynomial f on SM, we can "compress" it back to a symmetric tensor on M, denoted by f^{tens}. I like to refer to $(\cdot)^{\text{tens}}$ as **tensorization**.
- example: the general 2-tensor on a 2-dimensional manifold lifts to the polynomial $T_{11}v_1^2 + (T_{12} + T_{21})v_1 v_2 + T_{22}v_2^2$; hence the homogeneous polynomial $f(x)v_1^2 + g(x)v_1 v_2 + h(x)v_2^2$ corresponds to the symmetric tensor $T_{11} = f, T_{12} = T_{21} = \frac{1}{2}g, T_{22} = h$

B.2 "Polynomials" on SM

- a polynomial on SM is just the restriction to SM of a polynomial on TM
- being polynomial on SM is equivalent to being finite-degree (w.r.t. the basis of vertical spherical harmonics) and smooth
- just like on $\mathbb{R}^n$, Ω_m are exactly the restrictions to SM of **vertical solid harmonics** – homogeneous polynomials on TM that are harmonic on each tangent space
- The tensors that lift to vertical solid harmonics under $\tilde{\cdot}$ are called **trace-free**, and are denoted by Θ_m. This is addressed in [Sha94] and [GPSU16]. They use λ for the lifting map when restricted to this domain [PSU15].

B.3 Decomposition of X into X_+ and X_-

Theorem (Decomposition of X):

(i) For $u \in \Omega_m$, $Xu \in \Omega_{m-1} \oplus \Omega_{m+1}$ (Prop GK80b-3.2, p.164; stated in PS18 at bottom of p.11)

(ii) X_+ is injective on each Ω_m (Thm GPSU16-5.2, p. 30; stated in PS18 on p.16)

(iii) $X_+^* = -X_-$ (Lemma PSU15-3.1, p.14)

(iv) $\Omega_m = X_+\Omega_{m-1} \oplus \ker X_-$ (Lemma PSU15-3.2, p.14)

(v) In terms of tensors, the components of X act as follows (PSU15, p.15):

$$X_+ \cong_{\text{lift}} \text{proj}_{\Theta_{m+1}} d$$
$$X_- \cong_{\text{lift}} c_{m,d} \cdot \delta, \text{ where } \delta = -d^*$$

Remarks:

- all of these hold for for X^A as well
- (i) is in stark contrast with a fact we're very familiar with: in the familiar setting of $\mathbb{R}^n$, we know that taking any directional derivative of a polynomial just knocks the degree down by 1. Here, though, the theorem tells us that when we take a particular directional derivative, we also get some terms of *greater* degree
- [GK80] proves (1) by mostly algebraic considerations, so it doesn't really give any insight into what feature of the geometry causes the higher-degree terms to emerge
- note in particular that for $m = 0$, we have $X = X_+$

B.4 The function space $\mathcal{Z}$ and the Bundle N

To be able to formulate an energy identity for $\overset{\mathrm{v}}{\nabla} X$, we need norms on the domain and codomain. The domain is $C^\infty(SM)$ – for which we have a norm (namely, the $L^2(SM)$ norm), but we need to nail down a codomain and associated norm. The codomain will be the following function space:

Definition. The function space $\mathcal{Z}$ consists of smooth functions $Z : SM \to TM$ that satisfy the following properties:
(i) $Z(x, \underline{v}) \in T_x M$
(ii) $Z(x, \underline{v}) \perp \underline{v}$

To put an inner product on $\mathcal{Z}$, we'll realize it as sections of a bundle N, and use the bundle metric to obtain our inner product.

Construction of N

1. Over each point of $S_x M$ put a copy of $T_x M$ (i.e. right now all fibres over $S_x M$ are identical)
2. Chop each one down to a (different) hyperplane. The copy over $\theta = (x, \underline{v}) \in S_x M$ gets chopped down to $\underline{v}^\perp \subset T_x M$. I call this bundle N^{pre}.
3. Notice that this inherits a natural bundle metric:

$$\left\langle \underline{\xi}, \underline{\eta} \right\rangle_{N^{\text{pre}}_\theta} = \left\langle \underline{\xi}, \underline{\eta} \right\rangle_{T_x M}$$

4. Complexify (see C.1.1) to get N. (At this point, its a complex vector bundle over a smooth manifold.)
5. Since the metric on N^{pre} is a smooth family of real inner products, it too, can be complexified. This gives a Hermitian metric on N.

Definition. Under the identification $\mathcal{Z} \cong \Gamma(N)$, we equip $\mathcal{Z}$ with an inner product as follows:

$$(Z_1, Z_2)_{\mathcal{Z}} := \int_{SM} \langle Z_1|_\theta, Z_2|_\theta \rangle_{N_\theta} \, d\mu_L(\theta)$$

where μ_L is the Liouville metric on SM (see C.2.10)

- we use round brackets for this inner product; that's to keep it distinct from Riemannian metrics, for which we usually use angled brackets.

Appendix C

Background Knowledge

C.1 Algebra

C.1.1 Complexification of Vector Spaces

Definition. Given a vector space V over $\mathbb{R}$, its complexification $V^{\mathbb{C}}$ consists of:

- underlying set: $V \otimes_{\mathbb{R}} \mathbb{C}$ (looking at $\mathbb{C}$ as a 2D vector space over $\mathbb{R}$)
- multiplication by a complex scalar: $\alpha(\underline{v} \otimes \beta) \coloneqq \underline{v} \otimes \alpha\beta$

The dimension of $V^{\mathbb{C}}$ is:

- as a real vector space – $2\dim(V)$
- as a complex vector space – $\dim(V)$

As a real vector space, $V^{\mathbb{C}} \cong V \oplus V$, and we write it as $V + iV$ in the same way that we write $\mathbb{C}$ as $\mathbb{R} + i\mathbb{R}$. Under this identification, the scalar multiplication can be written $(a + bi)(\underline{v} + i\underline{w}) = (a\underline{v} - b\underline{w}) + i(b\underline{v} + a\underline{w})$

There's a natural way to realize V as a (real) subspace of $V^{\mathbb{C}}$ (considered here as a real vector space): the element corresponding to $\underline{v} \in V$ is just $\underline{v} \otimes 1$

If V has an inner product, that too can be complexified, so that entire inner product spaces can be complexified.

Reference: Wikipedia, *Complexification*

C.2 Geometry

C.2.1 Riemannian Manifolds with Boundary

The analysis of geodesics in manifolds with boundary turns out to be a very thorny subject, and it seems like not much has been written on the topic. The papers that I found are:

- *Geodesics in Riemannian manifolds-with-boundary*, Alexander and Alexander, 1981
- *The Riemann obstacle problem* by Alexander, Berg, Bishop, 1985
- *Cauchy uniqueness in the Riemannian obstacle problem*, Alexander, Berg, Bishop, 1986
- *Geodesics in Euclidean space with analytic obstacle*, Albrecht and Berg, 1991

Starting from the definition that geodesics are locally distance-minimizing, we lose smoothness right away; such curves can go back and forth between the boundary and the interior with a hard corner at each transition. So in fact the best we can hope for in general is C^1. Moreover, we lose uniqueness.

Typically, when we work with geodesics (in the non-boundary case), we use the PDE $\nabla_{\dot\gamma}\dot\gamma = 0$, but the appropriate PDE that reflects the situation in the boundary case is not so nice:

$$\nabla_{\dot\gamma}\dot\gamma = \begin{cases} 0 & \text{on } M^\circ \\ k(x)\underline{\nu} & \text{on } \partial M \ (\underline{\nu} \text{ is the outward unit normal}) \end{cases}$$

Moreover, the PDE must be solved in H^2, since the LHS won't even be defined at all points of the solution curve.

In the context of the problem we're dealing with in this report, this state of affairs raises the following questions:

- does/should $\tau(\theta)$ refer to the time at which γ_θ *arrives* at the boundary, or the time at which it *"exits"* the boundary (i.e. ceases to exist in the manifold)?
- if geodesics aren't unique, how do we decide which one to follow in computing the ray transform?

The good news is that all of these questions are circumvented by imposing strict convexity, which is a hypothesis for CDRM's.

Definition. A manifold with boundary has **strictly convex boundary** if the second fundamental form is positive definite.

Theorem. If a manifold has strictly convex boundary, then geodesics only intersect the boundary transversely. In particular, geodesics don't travel along the boundary, nor can they "kiss" the boundary as they travel.

Proof. On p.112 of [Sha94], it's recorded that $\partial_+\tau_+, \partial_-\tau_- \equiv 0$ (∂_+, ∂_- are defined opposite to ours, so the $+$'s and $-$'s appear "crossed"). Now, $\partial_+SM \cap \partial_-SM = \partial_0 SM$, which is the set of all boundary directions, and on this set τ_+, τ_- vanish simultaneously. This means that any "geodesic" starting at a boundary point exits the manifold exactly at that point going both forwards and backwards, so it reduces to a point and hence it's not a geodesic at all. ∎

Remarks

- This isn't so much a proof as it is a justification for how I know the theorem to be true. I'm assuming that the proof of the facts about $\tau_\pm$ involves showing that the "geodesics" along the boundary are just points, and so the reasoning in my "proof" is circular. I find it interesting that the results about $\tau_\pm$ are just slipped in the text as if they're trivial, because I think there's a little more going on than meets the eye.
- Here's an easy example that offers a heuristic proof: compute the second fundamental forms of $M = \bar{B}^2$ and $M = \mathbb{R}^2 \setminus B^2$, both of which have boundary S^1; you'll find that the former is identically 1 – "the boundary is as convex as possible" – and the later is identically -1 – "as far from convex as possible". Notice that in the former case, geodesics behave exactly as we're used to, but in the latter case the shortest path between two points on opposite sides of the removed disc will include an arc of S^1. This kind of bridges the gap between our intuitive understanding of convexity and the notion of convexity defined via the second fundamental form. So it feels like if a manifold is convex (according to the second fundamental form) near its boundary, then geodesics should behave similar to those in $\bar{B}^2$ (which adheres to our intuitive understanding of convexity and is *also* convex according to the second fundamental form)

C.2.2 Ricci Calculus

Placeholder

C.2.3 Solenoidal/Potential Decomposition of Tensor Fields

Definition. For symmetric (covariant) tensor fields on compact manifolds with boundary, we make the following definitions:

(i) A field T is called **solenoidal** if $\delta T \equiv 0$ ($-\delta$ is the adjoint of the inner derivative d)

(ii) A field T is called **potential** if it's the inner derivative of another symmetric tensor that vanishes on the boundary. That is, $T = dS$ for S with $S|_{\partial M} \equiv 0$

Theorem. Every symmetric (covariant) tensor field on compact manifold with boundary decomposes uniquely into solenoidal and potential components.

The proof is long and complex, so I'll just make some remarks about it:

- One can consider the analogous result for tensors on $\mathbb{R}^n$ as a "baby version". This is the content of §2.6 in [Sha94]. The proof in this case is much simpler; its done using the Fourier transform.
- The proof of the "full version" is all PDE's/hard analysis. It's the content of §3.3 in [Sha94]
- moreover, the proof the full version rests on a "theorem on normal solvabitiy" from a paper that's only available in Russian (*Solubility of boundary problems for general elliptic systems* by an L.R. Volevic, 1965) for existence and uniqueness

Reference: [Sha94], §2.4 (pp.39-42), §3.3 (pp.87-92)

C.2.4 Bundle-valued Forms

References:

- [Lee09] §8.5 (pp.370-373)

C.2.5 Connections

Part of the reason why I think the subject of connections was so hard to learn is that most of the things that you read are geared towards smooth vector bundles, where there are a whole bunch of interrelated notions at play (by contrast, in more general settings, fewer and fewer of these notions are even a thing):

- (linear) Ehresmann connection
- horizontal bundle
- horizontal lift
- parallel transport
- connection map
- Koszul connection
- covariant derivative

Moreover, any one determines all the others, so the lines between them are are bit blurred, and I think every author has their own unique concept/philosophy of connections, differentiated by how and in what measure these components feature.

The resources that really helped me to get a feeling for connections were:

- [Gol08] - this is pretty much exactly what I would've written here if it didn't already exist
- [Lee09], ch.12
- Wikipedia, *Connection (vector bundle)*
- Wikipedia, *Ehresmann connection*
- Wikipedia, *Connection (mathematics)*
- The page "Category:Connection (mathematics)" on Wikipedia contains links to many more articles

Notation used in this section

- $\mathcal{E}$ – vector bundle over M with total space E
- β – basis/frame field (local): $\beta = \{\underline{e}_1, \cdots, \underline{e}_r\}$
- σ – section of $\mathcal{E}$ (local); $\sigma = \sum {}^{\beta}s^i \underline{e}_i$
- $[\sigma]_\beta$ – coordinate vector of σ w.r.t. the basis β: $[\sigma]_\beta = \begin{bmatrix} {}^{\beta}s^1 \\ \vdots \\ {}^{\beta}s^r \end{bmatrix}$
- Ω – $\mathcal{E}$-valued form; $\Omega = \sum \underline{e}_i \otimes {}^{\beta}\omega^i$
- $[\Omega]_\beta$ – coordinate vector of Ω w.r.t. basis β: $[\Omega]_\beta = \begin{bmatrix} {}^{\beta}\omega^1 \\ \vdots \\ {}^{\beta}\omega^r \end{bmatrix}$

Koszul Connection

This type of connection is designed specifically for smooth vector bundles. It can either be formulated as a map from sections to bundle-valued 1-forms $\Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ or as a taking a vector field and section and returning a section $\mathfrak{X}(M) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ with specified properties in either case. From my reading, it doesn't seem like there's a consistent conceptual distinction between the terms "Koszul connection" and "covariant derivative", though Husemoller (p.284) defines a covariant derivative as map taking in a vector field and producing an endomorphism on sections: $\mathfrak{X}(M) \to \mathrm{End}\left(\Gamma(\mathcal{E})\right)$

Connection form (of a Koszul connection)

The connection form is a matrix A of 1-forms that encodes the action of the connection. This matrix is only defined in coordinates and it changes depending on the basis chosen. Its entries are defined by $\nabla \underline{e}_j = \sum \underline{e}_i \otimes {}^{\beta}A^i_j$. The fundamental identity here is $\nabla^{\mathcal{E}} = d + A$, although this is best thought of as a shorthand, with the proper interpretation being:

$$\left[\nabla^{\mathcal{E}}\sigma\right]_\beta = d^{(M)}\left[\sigma\right]_\beta + {}^{\beta}A \cdot [\sigma]_\beta$$

where:
- d is the exterior derivative (in contrast to the rest of this report, where it's the inner derivative) and it acts coordinate-wise
- the dot represents matrix multiplication

Basically, this allows us to compute the action of the connection in terms of two things we know how to do easily: exterior differentiation and matrix multiplication. Note: this identity only holds on 0-forms (i.e. sections of the bundle). For (bundle-valued) forms of higher degree, it's a bit more complicated; see "Covariant exterior derivative" below.

Change-of-basis

If the two basis fields are related by $\gamma = \beta \cdot P$, then the expression of the connection form in the new basis is:

$${}^{\gamma}A = P^{-1} \cdot dP + P^{-1} \cdot {}^{\beta}A \cdot P$$

Curvature form (of a Koszul connection)

The curvature form is a matrix of 2-forms that encodes the curvature of the connection. It's defined as $F^{\mathcal{E}} := d^{\beta}A + {}^{\beta}A \wedge {}^{\beta}A$ for *any* β (it turns out to be independent of coordinates). It's often summarized by dropping β, hence: $F^{\mathcal{E}} = dA + A \wedge A$. In this expression, d is again the exterior derivative, acting entry-wise, and the wedge represents regular matrix multiplication where the product between entries is the wedge product.

Covariant exterior derivative

The covariant exterior derivative is the operator $d^{\mathcal{E}} := d^{(M)} + A$. (just like the identity previously, this definition is also properly interpreted in coordinates: $\left[d^{\mathcal{E}}\Omega\right]_\beta = d^{(M)}\left[\Omega\right]_\beta + {}^{\beta}A \wedge [\Omega]_\beta$.) In contrast with the regular exterior derivative, the square of the covariant exterior derivative $d^{\mathcal{E}}$ is not necessarily 0. In fact, $\left(d^{\mathcal{E}}\right)^2 = F \wedge [-]$:

$$\begin{aligned}
\left(d^{\mathcal{E}}\right)^2 \sigma &= \left(d + {}^{\beta}A\right)\left(d\sigma + {}^{\beta}A\sigma\right) \\
&= d^2\sigma + d\left({}^{\beta}A \wedge \sigma\right) + {}^{\beta}A \wedge d\sigma + {}^{\beta}A \wedge \left({}^{\beta}A \wedge \sigma\right) \\
&= 0 + d^{\beta}A \wedge \sigma - {}^{\beta}A \wedge d\sigma + {}^{\beta}A \wedge \sigma + \left({}^{\beta}A \wedge {}^{\beta}A\right) \wedge \sigma \\
&= \left(d^{\beta}A\right) \wedge \sigma + \left({}^{\beta}A \wedge {}^{\beta}A\right) \wedge \sigma \\
&= \left(d^{\beta}A + {}^{\beta}A \wedge {}^{\beta}A\right) \wedge \sigma \\
&= F \wedge \sigma
\end{aligned}$$

The relationship between the connection and the covariant exterior derivative on forms of general degree is $d^{\mathcal{E}} = c_{\text{degree}}\text{Alt}\nabla$. Notice that since "alternationization" isn't injective on forms of degree > 1, this can't be used to compute the action of the connection on such forms as it could for sections.

Connector/Connection map

To motivate the existence of the connection map, consider this: the primary purpose of connections on smooth vector bundles is to furnish a notion of covariant derivative for sections. The "first" notion of derivative - the differential/tangent map, takes values in TE, whereas we'd like it to take values in the same space as the original function (that space being E). The connection map exists to do just that: $\mathcal{K}$ is a map from TE to the bundle E, and the exact way that it does so is chosen so that the induced differentiation operator is covariant.

There are various ways of defining it depending on what notion of connection you start with. If you start with an Ehresmann connection (which is just a declaration of horizontal bundle), then the connection map takes a vector, projects it onto the vertical bundle (along the horizontal bundle), and then maps the projection down into E via the canonical isomorphism. To give a taste of how this gives a covariant derivative: when defining a covariant derivative from an Ehresmann connection, the idea is that we have decided that any sections whose derivative is horizontal are the ones we'll consider "constant" (parallel). Say you define the covariant derivative as the tangent map followed by the connection map. The tangent map gives you the velocity vector field of the section; say it's horizontal. When you apply the connection map, all the horizontal vectors get projected to the zero vector in the vertical section, which obviously gets brought down to the zero section in E. So what this shows is that with this definition of covariant derivative, sections that we declared as"constant" do have derivative 0. This way of defining the connection map is found in [Lee09]

[Pat99] does case where you start with Koszul connection or jump straight in with an "axiomatic" covariant derivative. His presentation only covers the particular case where E is the tangent bundle, but it does generalize verbatim by just replacing certain instances of "TM" by "E".

References

- [Lee09], ch. 12
- [Gol08]

C.2.6 The Riemann Curvature Tensor

Theorem (Symmetries of R):

(i) $R_{ijkl} = -R_{jikl}$ (anti-symmetry in first two indices)
(ii) $R_{sskl} = 0$ (corollary of previous)
(iii) $R_{ijkl} = -R_{ijlk}$ (anti-symmetry in last two indices)
(iv) $R_{ijss} = 0$ (corollary of previous)
(v) $R_{ijkl} = -R_{klij}$
(vi) $R_{ijks} + R_{jkis} + R_{kijs} = 0$ (First Bianchi Identity)
- It doesn't have to be the last index held constant it's true anytime you hold one index fixed and cyclically permute the other three

References:

- [dC06], Ch.4 (p.98)
- Wikipedia, *Riemann curvature tensor*

C.2.7 Geodesic Vector Field

Reference: [Pat99]

C.2.8 "Vertical" and "Horizontal"

The terms "vertical" and "horizontal" come from the theory of smooth fibre bundles. A smooth fibre bundle is specified by four pieces of data: The base space M, the total space E, the fibre F, and the projection $\pi : E \to M$.

Of all the objects with "vertical" or "horizontal" in their name, the first one that gets defined is the **vertical bundle**. This is the collection of vectors in TE that are tangent to the fibres. Denoted VE. Notice that it's a second-order bundle: that is, it doesn't consist of elements of the bundle E, but rather elements of the tangent bundle *over* E.

Once VE is defined, it's possible to talk about horizontal bundles, which are defined in reference to the vertical bundle. Notice the plural: there isn't just one. A sub-bundle of TE is **horizontal** if each fibre is complementary to the vertical bundle. That is, $E_x = H_x \oplus V_x$.

In practice, however, we usually have a way of declaring a canonical horizontal bundle: in essence, that's exactly what connections do. In particular, we have a canonical horizontal bundle on (the tangent bundle of) Riemannian manifolds, because the metric on M induces a metric on TM (the Sasaki metric), which in turn induces a connection on TTM (the Levi-Civita connection).

Other objects:

- **vertical/horizontal vector field** is a vector field on E that consists only of vectors in the vertical/horizontal bundle
- the **vertical derivative** and **horizontal derivative** of a smooth function $\overset{\text{v}}{\nabla} u, \overset{\text{h}}{\nabla} u$ are the components of the gradient ∇u with respect to the decomposition $TE = VE \oplus HE$

Some facts about the vertical and horizontal bundles over a *vector* bundle (with connection)

- $VE = \ker d\pi_E$
- $HE = \ker \mathcal{K}$ (the connection map)
- There's a natural isomorphism-along-π_E from $VE \to E$. It's just identification of $T_e E_p \cong E_p$.
- $d\pi_E$ restricts to an isomorphism-along-π_E from $HE \to E$
- $\mathcal{K}$ restricts to an isomorphism-along-π_{TM} from $VE \to E$
- $(d\pi_E, \mathcal{K}) : TE \to E \times TM$ is an isomorphism along (π_E, π_{TM})
- In the case that M is a Riemannian manifold and $E = TM$ (with the Sasaki metric), the isomorphisms are isometries (simply because the Sasaki metric is defined to make that happen)

References:

- [Pat99], §1.3.1 (pp.11-13)
- [Lee09], pp.511-512, 518, 520
- Wikipedia, *Vertical and horizontal bundles*

C.2.9 Sasaki Metric

The Sasaki metric is a natural metric on the tangent bundle – in our case, restricted to the sphere bundle – of a Riemannian manifold, induced by the metric on the base manifold. Whereas the notations g and $\langle -, - \rangle$ are used for the metric on M, the notations $\hat{g}$ and $\langle\langle -, - \rangle\rangle$ are used for the metric on TM.

Since the vertical and horizontal bundles are each isomorphic (along π_{TM} to TM – the vertical bundle via the connection map $\mathcal{K}$ and the horizontal bundle via $d\pi_{TM}$, it's natural to metrize them so that the isomorphisms become isometries (which is done by pulling the metric on TM back by the corresponding isomorphism). The Sasaki metric is the sum of these two metrics. In symbols:

$$\left\langle\left\langle \underline{\xi}, \underline{\eta} \right\rangle\right\rangle_\theta := \left\langle d_\theta \pi(\underline{\xi}^{\mathrm{h}}), d_\theta \pi(\underline{\eta}^{\mathrm{h}}) \right\rangle_x + \left\langle \mathcal{K}_\theta(\underline{\xi}^{\mathrm{v}}), \mathcal{K}_\theta(\underline{\eta}^{\mathrm{v}}) \right\rangle_x$$

Note: since the kernels of $d\pi$ and $\mathcal{K}$ are exactly the co-kernel of the other, there's no "double-counting". Also, note that this definition makes the vertical and horizontal bundles mutually orthogonal.

References

- [Pat99], p.13
- Wikipedia, *Sasaki metric*

C.2.10 Liouville Measure on SM

The Liouville measure arises from the canonical contact form.

Definition:

(i) The **contact form** on the tangent bundle of a Riemannian manifold is:

$$\alpha_\theta(\xi) \coloneqq \langle\!\langle \xi, X|_\theta \rangle\!\rangle$$

(ii) The **Liouville measure** on SM is

$$\mu_L(E) \coloneqq \int_E \alpha \wedge (d\alpha)^{\wedge(d-1)}$$

Proposition. It turns out that the form used to define the Liouville metric is just a multiple of the volume form on SM that comes from the Sasaki metric:

$$\alpha \wedge (d\alpha)^{\wedge(d-1)} = \pm(d-1)\text{vol}_{SM}$$

Hence the Liouville volume of any set is just $d-1$ times its "intrinsic" volume

- **Note:** the LHS is defined on all of TM, while the RHS is only defined on SM

Reference: [Pat99], §1.3.3 (pp.15-18)

C.3 PDE's

C.3.1 Transport Equation

The transport equation is a first order PDE, and so it can be solved by the method of characteristics.

Reference: Evans, §3.2 (pp.96-114)

C.4 Harmonic Analysis

C.4.1 Generalities

Using the eigenvalues of the Laplacian on the given space as an orthonormal basis for L^2.

C.4.2 Spherical Harmonics

Classical theory

Spherical harmonics can be thought of in several ways.

- they are **not**, as the name suggests, harmonic functions on the sphere
- they **are**, however, eigenfunctions for the Laplace-Beltrami operator on S^n.
- they arise as the angular parts of separated solutions to the Laplace equation on $\mathbb{R}^n$
- there's no closed-form expression; they are most easily expressed in terms of "associated Legendre polynomials", which are themselves expressed in terms of the (basic) Legendre polynomials, which are themselves expressed recursively
- they can also be characterized as the restriction of homogeneous, harmonic (on $\mathbb{R}^{n+1}$) polynomials – called (regular) **solid harmonics** – to the sphere, which is where the name comes from.
- in fact, restriction is a **bijection** from homogeneous, harmonic polynomials on $\mathbb{R}^{n+1}$ to the eigenspaces of the spherical Laplacian. The inverse is "extension by homogeneity"

Homogeneous polynomials of a given degree can be decomposed as follows:

$$\mathscr{P}_m = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} r^k \mathscr{A}_{m-2k}$$

$$\mathscr{P}'_m = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \mathscr{H}_{m-2k}$$

where (using notation of [SW71]):

- $\mathscr{P}_m$ - homogeneous polynomials of degree m on $\mathbb{R}^n$
- $\mathscr{P}'_m$ - restrictions to S^{n-1} (my notation)
- $\mathscr{A}_j$ - **harmonic** homogeneous polynomials on $\mathbb{R}^n$ (of degree j) ((solid harmonics))
- $\mathscr{H}_j$ - spherical harmonics of degree j – these are the restrictions of $\mathscr{A}_j$ to S^{n-1}, so we can think of $\mathscr{H}_j = \mathscr{A}'_j$

Bases (real version; not normalized)

m	$Y_{l,m}$			
-3				$y(3x^2 - y^2)$
-2			xy	xyz
-1		y	yz	$y(4z^2 - x^2 - y^2)$
0	1	z	$2z^2 - x^2 - y^2$	$z(2z^2 - 3x^2 - 3y^2)$
1		x	xz	$x(4z^2 - x^2 - y^2)$
2			$x^2 - y^2$	$z(x^2 - y^2)$
3				$x(x^2 - 3y^2)$

Examples

$$x^2 = \underbrace{\tfrac{-1}{6}Y_{20} + \tfrac{1}{2}Y_{22}}_{\in \mathscr{A}_2} + r^2 \cdot \underbrace{\tfrac{1}{3}Y_{00}}_{\in \mathscr{A}_0} \qquad = \tfrac{2}{3}x^2 - \tfrac{1}{3}y^2 - \tfrac{1}{3}z^2 + r^2 \cdot \tfrac{1}{3}$$

$$x^3 = \underbrace{\tfrac{-3}{20}Y_{31} + \tfrac{1}{4}Y_{33}}_{\in \mathscr{A}_3} + r^2 \cdot \underbrace{\tfrac{3}{5}Y_{11}}_{\in \mathscr{A}_1} \qquad = \tfrac{3}{20}x^3 - \tfrac{3}{5}xy^2 - \tfrac{3}{5}xz^2 + \tfrac{1}{4}x^2 + r^2 \cdot \tfrac{3}{5}x$$

Spherical harmonics are useful in investigating the Fourier transform on $\mathbb{R}^n$; while the connection is irrelevant for the present paper, info on how they are used can be found in [SW71] Ch.4.

On sphere bundles

Lemma 5.4 (Vertical derivatives of vertical spherical harmonics). Consider a vertical spherical harmonic $u \in \Omega_m$. $\partial_j : C^\infty(SM) \to C^\infty(SM)$ is a modified version of the directional derivative operator $\partial_{y_j} : C^\infty(TM) \to C^\infty(TM)$ (see A.1, Vector fields and other tensors). We have:

(i) $\partial_j[u] \in \Omega_{m+1} \oplus \Omega_{m-1}$, where the Ω_{m+1}-part is given by $-mv_j u$.
(ii) Denote the Ω_{m-1}-part by h_j. For any i, j, we have $v_i h_j \in \Omega_m \oplus \Omega_{m-2}$
(iii) The Ω_{m-2}-part of $v_i h_j$ is symmetric in the sense that $v_i h_j$ and $v_j h_i$ have the same Ω_{m-2}-part.

The Ω_{m-2} part of $v_i h_j$ is denoted h_{ij}, and the Ω_m-part is denoted f_{ij}. For easy reference, we record the expansions here:

$$\begin{aligned} \partial_j[u] &= -mv_j + h_j \\ v_i h_j &= h_{ij} + f_{ij} \\ h_{ij} &= h_{ji} \end{aligned}$$

All of this still holds if the indices are raised

- used in the proof of Lemma 5.3(i)

Lemma (Decomposition of $L^2(SM)$ into vertical spherical harmonics).

$$L^2(SM) = \bigoplus_{\mathbb{N}} H_m(SM)$$

In the course of proving Lemma 4.3, we find an expression that encodes the eigenvalues of the vertical Laplacian in terms of the dimension of the manifold (p.19):

$$\lambda_l \left(1 - 1/l\right) \left(1 + \frac{1}{l+d-2}\right) = \lambda_l - (d-1)$$

References:

- [Gar11]
- [Shu01] §22
- [SW71] Ch IV.2 to approx. p.141
- [GK80] §2,3 to approx. p. 164
- [PSU15] p.15
- Wikipedia, *Spherical harmonics*
- Wikipedia, *Table of spherical harmonics*

C.5 Measure Theory

C.5.1 Fubini's Theorem on Bundles

Placeholder

C.5.2 Disintegration

Reference: Wikipedia, *Disintegration theorem*

Bibliography

[dC06] Manfredo P. do Carmo. *Riemannian Geometry*. Birkhäuser, 2006.

[Gar11] Paul Garrett. Harmonic analysis on spheres, 2011.

[GK80] Victor Guillemin and David Kazhdan. Some inverse spectral results for negatively curved n-manifolds. *Proceedings of Symposia in Pure Mathematics*, 36:153–180, 1980.

[Gol08] Timothy E. Goldberg. What is a connection, and what is it good for?, 2008.

[GPSU16] Colin Guillarmou, Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The X-Ray Transform for Connections in Negative Curvature. *Communications in Mathematical Physics*, 343(1):83–127, 2016.

[Lee09] Jeffrey M. Lee. *Manifolds and Differential Geometry*. Graduate Studies in Mathematics, vol. 107. AMS, 2009.

[Pat99] Gabriel P. Paternain. *Geodesic Flows*. Birkhäuser, 1999.

[PS88] Leonid Pestov and Vladimir A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Siberian Mathematical Journal*, 29(3):427–441, 1988.

[PS18] Gabriel P. Paternain and Mikko Salo. Carleman estimates for geodesic X-ray transforms. 2018.

[PSU12] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The attenuated ray transform for connections and Higgs fields. *Geometric and Functional Analysis*, 22(5):1460–1489, 2012.

[PSU13] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography on surfaces. *Inventiones Mathematicae*, 193(1):229–247, 2013.

[PSU15] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Mathematische Annalen*, 363(1-2):305–362, 4 2015.

[PU05] Leonid Pestov and Gunther Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Annals of Mathematics*, 161:1093–1110, 2005.

[Sha94] Vladimir A. Sharafutdinov. *Integral Geometry of Tensor Fields.* Inverse and Ill-Posed Problems. De Gruyter, 1994.

[Shu01] Mikhail A. Shubin. *Pseudodifferential Operators and Spectral Theory.* Springer, 2nd edition, 2001.

[SW71] Elias M. Stein and Guido L. Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton Mathematical Series. Princeton University Press, 1971.